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Discrete Time Gathering of Agents with Bearing Only and Limited Visibility Range Sensors

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Abstract

We analyze a gathering process for a group of mobile robotic agents, identical and indistinguishable, with no memory (oblivious) and no common frame of reference (neither absolute location nor a common orientation). The agents are assumed to have bearing only sensing within a limited visibility range. We prove that such robots can gather to a small disk in the $\mathbb{R}^2$-plane within a finite expected number of time-steps, implementing a randomized visibility preserving motion law. In addition, we analyze the dynamics of the cluster of agents after gathering, and show that the agent-cluster preforms a random-walk in the plane.

1 Introduction

Gathering is a basic task in multi-agent systems and a lot of research is devoted to the development of algorithm for accomplishing it, under various assumptions on agents' motion and sensing capabilities. Here we address the problem of achieving gathering with oblivious, anonymous (identical and indistinguishable) and non-communicating robots, lacking of a joint frame of reference in space that are capable of sensing their neighbours' bearing only within a limited visibility range.

The problem was addressed in a discrete time framework in Gordon & Bruckstein [1][2]. The main result, proved in [2], is that a randomized rule of motion according to which each agent jumps a limited distance (σ) to a random location inside a region determined by the neighbours' directions, achieves both cohesion of the swarm and gathering to a constellation within a disk of diameter equal to the visibility range (V), in finite expected time. Experimentally it was observed that, due to bearing only sensing, the agents' cluster was of a smaller size, of the order of the step size σ, and was drifting in the plane in what seemed to be a random walk.

In this paper, we modify the motion law discussed in [2], and prove that it gathers the agents of the system to a disk with a radius equal to the agents' maximal step size σ within a finite expected number of time steps. Furthermore, we prove that as time tends to infinity the distribution of the agents' centroid converges in probability to a distribution of a random-walk variable.

We start by presenting the dynamics of a multi-agent system of agents having unlimited visibility, applying a non-randomized motion-law. We proceed with an explanation of the necessity for randomization in the motion law, when applying it to a system with agents having limited-visibility. Then, we formally define the adjusted randomized motion law, and prove that it gathers the agents to a disk with a radius equal to the length of the maximal allowed step, σ. We end this paper proving that the distribution function of the average position of the agents preforms a random motion converging in probability to the distribution of a random-walk.

Note that by "gathering", we mean that the system actually reaches the goal state, which is a closely clustered constellation of agent locations, within a finite time or number of time-steps, while the meaning of "converging" implies an asymptomatic approach to the goal state as time progresses, without necessarily reaching it within a finite time.

2 Preliminaries

We consider a system of n identical, anonymous, and oblivious agents in the $\mathbb{R}^2$ plane specified by their time varying locations $\{p_i(k)\}_{i=1,2,\ldots,n}$. We assume that the agents are able to sense the direction to their neighbours (i.e. bearing only sensing), so that their 'knowledge" about neighbours is partial, and their motions are determined by their current location and the set of unit vectors pointing to their neighbours. The neighbours are defined for each agent i at time-step k as a set of agents located within a given visibility range V form its position, $p_i(k)$, and are denoted by $N_i(k)$.

The neighbourhood relation between agents is usually described by a time dependent visibility-graph. Notice that when dealing with unlimited visibility, the set $N_i(k)$ comprises all the agents except i, and the visibility-graph is complete, i.e. all agents sense each other.

The proofs in this paper require the use of some results from basic geometry and the theory of random-processes which can be found in Appendix 1.

3 Unlimited visibility

The agents of the system are assumed to have bearing only sensing. Since, they can not estimate the distance to their neighbours. Agents can not determine the relative positions of their neighbours. However, an agent can readily figure out whether it is located at a corner of the convex-hull of the agents' constellation. The agents located at such corners "know" in which direction they should move in order to enter, or to go through the agent constellation's convex-hull. If they move in a good direction, they have a chance to decrease the convex-hull's area and perimeter. In the following section we use this ability of the agents to postulate a motion law which gathers the agents into a small region.

Since, only the agents located at the convex-hull corners can estimate a "good" direction of movement, we dictate that only they should move. An agent i is located at a convex-hull corner only if $\psi_i(k)$, the angle of the current minimal angular-sector anchored at agent i's position and containing all its neighbours, is less than π. We set the motion law for the agents as follows: an agent located at a convex-hull corner moves in the direction of $\hat{\psi}_i(k)$, the unit vector in the direction of the bisector of angle $\psi_i(k)$ a step size determined by a parameter and the cosine of $\psi_i(k)/2$, while agents inside the convex-hull stay put. Note that, in some cases the agents may cross the current convex-hull and leave it, to find themselves outside of it. In the sequel we consider the minimal enclosing circle of the agents constellation for analysing the system's dynamics, and do not attempt to rely on properties of the convex-hull.

We simplify the analysis of the swarm's dynamics by defining a motion law ensuring that the agents' average position is a system invariant.

The law of motion:
Each agent i located at a convex-hull's corner jumps in the direction of the unit vector $\hat{\psi}_i$ a distance proportional to $\cos(\psi_i(k)/2)$, i.e. half the sum of the unit vectors pointing from $p_i(k)$ to its extremal left and right neighbours. Considering that our agents are capable of jumping steps of length at most $\sigma > 0$, the new motion law for an agent i is

$$p_i(k+1) = p_i(k) + \begin{cases} \sigma\hat{\psi}_i(k)\cos(\frac{\psi(k)}{2}) & \psi_i(k) < \pi \\ 0 & o.w. \end{cases} \tag{1}$$

***Lemma* 1.** *In a multi-agent system with dynamics (1), $\bar{p}(k)$, the average position of the agents, is invariant.*

Proof. Let $U_i^+(k)$ and $U_i^-(k)$ be the unit vectors pointing from the position of agent i to of the current extremal left and right agents defining the current minimal angular-sector anchored at $p_i(k)$ and containing all its neighbours.

Let $CH(k)$ and $\partial CH(k)$ be the convex-hull of the agents' constellation and the set of agents located at its corners. Notice that for any agent $q \in \partial CH(k)$, located at a corner of the convex-hull, the associated unit vectors $U_q^+(k)$ and $U_q^-(k)$ are pointing to the next left and right corners of the convex-hull, and consequently the associated angle $\psi_q(k)$ is less then π. Furthermore, for each agent $\tilde{q} \notin \partial CH(k)$, we have that $\psi_{\tilde{q}(k)} \geq \pi$, hence it stays put.

Let us number the agents of $\partial CH(k)$ in an ascending order of indices choosing an arbitrary agent to be 1, so that the agent occupying the next left corner (in the clockwise direction) of the convex-hull is marked by 2 and so on. Then, the unit vectors $U_i^-(k)$ and $U_i^+(k)$ are pointing from $p_i(k)$ to $p_{i-1}(k)$ and $p_{i+1}(k)$ respectively. Hence, $U_i^+(k)$, the unit vector in the direction of the left border of an agent i's angular-sector, is directed opposite to the direction of $U^-_{i+1}(k)$, i.e.

$$U^-_{i+1}(k) = -U^+_i(k)$$

Rewriting the motion law (1) for an agent $i \in \partial CH(k)$ using the unit vectors $U^-_i(k)$ and $U^+_i(k)$, we have that

$$p_i(k+1) = p_i(k) + \sigma\hat{\psi}_i(k)\cos(\psi(k)/2) = p_i(k) + \sigma\frac{U^-_i(k) + U^+_i(k)}{2}$$

therefore since the agents $j \notin \partial CH(k)$ stay put, the average position of the agents at time step $k+1$ is

$$\bar{p}(k+1) = \frac{1}{n}\sum_i p(k+1) = \bar{p}(k) + \frac{\sigma}{n}\sum_{i\in\partial CH(k)}\frac{U^-_i(k) + U^+_i(k)}{2} =$$

$$= \bar{p}(k) + \frac{\sigma}{2n}\left(\sum_{i\in\partial CH(k)} U^-_i(k) - \sum_{i\in\partial CH(k)} U^-_i(k)\right) = \bar{p}(k) + 0$$

hence $p_i(k+1) = p_i(k)$ for any time-step k, proving Lemma 1.

□

We next prove that this system gathers to a disk of radius σ within a finite number of time-steps for any initial constellation $P(0)$. Our proof is based on the decrease rate of $R(k)$, the radius of the smallest enclosing circle of the agents. We shall show that, if $R(k)$ is greater than σ, it decreases to σ within a finite number of time-steps, and once it is equal to or smaller than σ, it remains that way. We do so by showing that an agent located at a distance of less than $\sigma/2$ from $C(k)$, the center of the smallest enclosing circle, will never jump to a location farther than σ from $C(k)$, and an agent located at a distance greater than or equal to $\sigma/2$ from $C(k)$ will never jump to a farther location from $C(k)$ (Lemma 2). In addition we show that, if $R(k)$ is greater than σ, at least two agents that lie on the circumference of that circle or within a close proximity to it will jump to locations closer to $C(P(k))$ by a strictly positive length (Lemma 3). Therefore, if $R(k)$ is greater than σ, it decreases in no more than $\lceil n/2\rceil$ time-steps by a quantity bounded away from zero by a constant.

In order to simplify our proof, without loss of generality, we let $C(k)$ to be the location of the origin of the $\mathbb{R}^2$-plane.

Lemma 2. *In a multi-agent system with dynamics (1), if $\|p_i(k)\|$, the current distance between the position of agent i and $C(k)$ is greater than or equal to $\sigma/2$, then at the next time step $\|p_i(k+1)\|$ will be less than or equal to $\|p_i(k)\|$. Otherwise, $\|p_i(k+1)\|$ will be less than or equal to σ.*

Proof. Let $\theta_i(k)$ be the angle between the movement direction of an agent $i \in \partial CH(k)$ and the vector pointing from $p_i(k)$ to $C(k)$. Let us divide the current minimal enclosing circle into two half circles by a line defined by the points $p_i(k)$ and $C(k)$ (the dashed line in Figure 1).

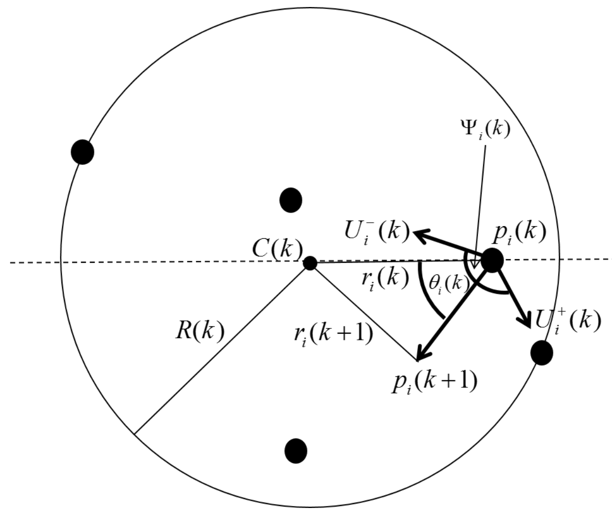

Figure 1: Minimal enclosing circle bipartition. The dashed line divide the current minimal enclosing circle into two half circles.

By Proposition 4 (See Apendix), there is at least one agent lying on each one of those half circles, therefore the angles $\theta_i(k)$ and $\psi_i(k)$, which associate with the movement of agent $i \in \partial CH(k)$, are bounded as follows:

$$0 \leq \theta_i(k) \leq \frac{\pi}{2} \quad \text{and} \quad 2\theta_i(k) \leq \psi_i(k) \leq \pi$$

Then, by the motion law (1) we have that,

$$\begin{aligned}
&\|p_i(k+1)\| = \\
&\sqrt{\|p_i(k)\|^2 + \left(\sigma \cos\left(\tfrac{\psi_i(k)}{2}\right)\right)^2 - 2\|p_i(k)\|\sigma \cos\left(\tfrac{\psi_i(k)}{2}\right)\cos(\theta_i(k))} \leq \\
&\sqrt{\|p_i(k)\|^2 + \sigma^2 \cos\left(\tfrac{\psi_i(k)}{2}\right)\cos(\theta_i(k)) - 2\|p_i(k)\|\sigma \cos\left(\tfrac{\psi_i(k)}{2}\right)\cos(\theta_i(k))} = \\
&\sqrt{\|p_i(k)\|^2 + \sigma \cos\left(\tfrac{\psi_i(k)}{2}\right)\cos(\theta_i(k))\left(\sigma - 2\|p_i(k)\|\right)}
\end{aligned} \tag{2}$$

Since, $\sigma \cos(\psi_i(k)/2)\cos(\theta_i(k)) \geq 0$, we have that if $\|p_i(k)\| \geq \sigma/2$, then

$$\|p_i(k+1)\| \leq \|p_i(k)\|$$

Otherwise (if $\|p_i(k)\| < \sigma/2$),

$$\|p_i(k+1)\| \leq \sqrt{\|p_i(k)\|^2 + \sigma \cos\left(\frac{\psi_i(k)}{2}\right)\cos(\theta_i(k))\left(\sigma - 2\|p_i(k)\|\right)} \leq$$

$$\leq \sqrt{\|p_i(k)\|^2 + \sigma\left(\sigma - 2\|p_i(k)\|\right)} = \left|\|p_i(k)\| - \sigma\right| \leq \sigma$$

Hence, $\|p_i(k+1)\|$ is bounded as claimed in Lemma 2.

□

***Lemma* 3.** *For a strictly positive $\delta < \sigma/2$, in a multi-agent system with dynamics (1), if $R(P(k))$ is greater than or equal to σ, then there are at least two agents located within the range δ from the circumference of the agent constellation's minimal enclosing circle that will jump to a distances closer to $C(P(k))$ by lengths bounded away from zero by a constant.*

Proof. By Proposition 5 in Appendix 1, we have that for a $\delta < R(P(k))$ there are at least two agents $s_{1,2}$ located within the range δ from the circumference of the minimal enclosing circle, and at different convex-hull corners with inner angles $\psi_{s_{1,2}}(k)$ bounded away bellow π by a constant as follows:

$$\psi_{s_{1,2}}(k) \leq \varphi(R(0),\delta) = \pi - \frac{2atan\left(\delta/\sqrt{R(0)^2-\delta^2}\right)}{m}$$

If $\|p_{s_{1,2}}(k)\| \geq \sigma$, then by (2) we have that $\|p_{s_{1,2}}(k+1)\|$ is bounded bellow $\|p_{s_{1,2}}(k)\|$ by a constant as follows:

$$\|p_{s_{1,2}}(k+1)\| =$$

$$\sqrt{\|p_{s_{1,2}}(k)\|^2 + \sigma\cos\left(\frac{\psi_{s_{1,2}}(k)}{2}\right)\cos(\theta_{s_{1,2}}(k))\left(\sigma - 2\|p_{s_{1,2}}(k)\|\right)} \leq$$

$$\sqrt{\|p_{s_{1,2}}(k)\|^2 - \sigma^2\cos^2\left(\frac{\varphi(R(0),\delta)}{2}\right)}$$

proving Lemma 3.

□

***Theorem* 1.** *A multi-agent system with dynamics (1) gathers to a disk of radius σ within finite number of time steps.*

Proof. By Lemma 2, no agent located at a distance greater than σ from $C(k)$ can jump to a farther distance, and we have that all agents located within a range of σ from $C(P(k))$ remains within this range at the next time-step. Furthermore, by Lemma 3, we have that if $R(P(k))$ is greater than σ, then there are at least two agents, located on the circumference of the smallest enclosing circle or at a distance less than δ from it jumping to positions closer to $C(P(k))$ by at least a constant quantity as discussed next. If $R(k)$ is greater than σ, after at most $\lceil n/2 \rceil$ consecutive time-steps all the agents of the system will fit into a smaller disk centered at $C(P(k))$ of a radius less than or equal to $\sqrt{\|R(k)\|^2 - \sigma^2\cos^2\left(\varphi(R(0),\delta)/2\right)}$, hence the radius of the minimal enclosing circle will decrease within every sequence of $\lceil n/2 \rceil$ time-steps by at least $\sigma\sin(\varphi(R(0),\delta)/2)$. As a consequence, all agents will gather to a disk of radius σ within a finite number of time-steps.

□

Recall that, by Lemma 1 the average position of the agents is invariant, therefore we may claim that the system gathers to a **static** disk of the radius 2σ centred at $\bar{p}$.

4 Limited visibility

We next assume that the agents have a limited visibility: an agent can "see" only agents located within its visibility range V. Unlike in the former section, which could "figure out" whether they are located at corners of the convex-hull or not, here the agents can not decide on this. However, despite the agents' lack of information, they will still be able to preform the basic task of preserving the visibility to their neighbours while moving, and hence they will be able to ensure connectivity, and under a randomized motion rule even ensure a monotone evolution of the visibility graph of the system toward a complete visibility graph.

We next present a motion rule, first suggested by Gordon at.el. in [2], which is a restriction on the regions the agents may move into, preventing them from losing visibility to their neighbours. An agent i may move only into the allowable region $AR_i(k)$, defined as follows:

Let $D_r(c)$ be a disc of radius r centered at point c, and let $c_{ij}(k)$ be a point at a distance $V/2$ from $p_i(k)$ in the relative direction of $p_j(k)$, i.e.

$$c_{ij}(k) = p_i(k) + \frac{V}{2}\frac{p_j(k) - p_i(k)}{\|p_j(k) - p_i(k)\|}$$

Then, the allowable region of an agent i with $N_i(k)$ as its current set of neighbours is

$$AR_i(k) \triangleq \left(\bigcap_{j\in N_i(k)} D_{\frac{V}{2}}(c_{ij}(k))\right) \cap D_{\frac{V}{2}}(p_i(k)) \tag{3}$$

see Figure 2.

Lemma 4. *If all agents move inside their allowable regions, none of them will lose visibility to its neighbours.*

Proof. Considering an agent i, we realize that if it "sees" an agent j in a given direction, agent j will be somewhere at a distance less than V from it. If the agent is at a distance V, then clearly both i and j can move into a disc of radius $V/2$ centered at their average location $(p_i(t) + p_j(t))/2$ without losing mutual visibility. If j will be at a distance less than V from i then they can again move into a disk of a radius $V/2$ centered at the average of their locations. Hence we have that the intersection of all possible moves for agent i, due to all the possible locations of agent j within $r < V$ distance from agent i, in the direction to j (known to i), is given by

$$AR_{ij}(k) = \bigcap_{r=o}^{V} D_{\frac{V}{2}}\left(p_i(k) + \frac{1}{2}\frac{p_j(k) - p_i(k)}{\|p_j(k) - p_i(k)\|}r\right) =$$

$$= D_{\frac{V}{2}}(p_i(k)) \bigcap D_{\frac{V}{2}}\left(p_i(k) + \frac{1}{2}\frac{p_j(k) - p_i(k)}{\|p_j(k) - p_i(k)\|}V\right) =$$

The allowable region for i to jump will be

$$AR_i(k) = \bigcap_{j\in N_i(k)} AR_{ij}(k)$$

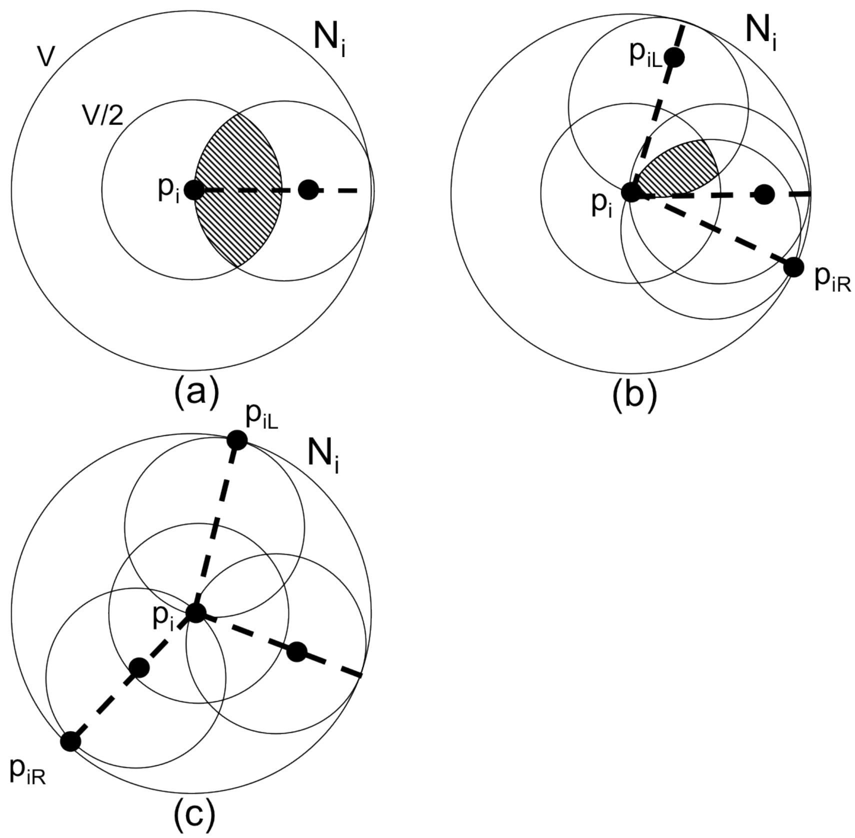

Figure 2: Allowable regions for agent i. (a) Single neighbour. (b) Intersection between the extreme left agent's disc, the extreme right agent's disc, and the disc $D_{\frac{V}{2}}(p_i(k))$. (c) No allowable region since the intersection yields an empty region.

hence we obtain formula (3).

Therefore, for any pair of neighbours i and j, if both i and j move into their allowable region, we have that $AR_i(k)$ and $AR_j(k)$ are contained in $D_{V/2}((p_i(k)+p_j(k))/2)$, hence the distance between them remains within V.

Note that if the agents of the set $N_i(k)$ surround the position of agent i (i.e. $\psi_i(k) > \pi$), its allowable region will shrink to a single point located at its position, hence it may not move without risking losing visibility with some of its neighbours.

□

From the above result it is clear that under the motion law (1), when $\sigma < V/2$, maintains the connectivity of the system's visibility graph, is maintained.

***Corollary* 1.** *Given that $\sigma \le V/2$, in a system where all agents move according to dynamic law (1), none of the agents lose visibility with their neighbours, hence the connectivity of the visibility graph preserved.*

Proof. For each agent i which currently sees all its neighbours in an angular section of angle $\psi_i(k) < \pi$, let $\theta_{ij}(k)$ be the angle between the vectors $\hat{\psi}_i(k)$ and $p_j(k) - p_i(k)$ where $j \in N_i(k)$ (see Figure 3), and let $Limit_i(k)$ and $Limit_{ij}(k)$ be the lengths of the section segments crossing $AR_i(k)$ and $AR_{ij}(k)$ respectively, starting at $p_i(k)$ in the direction of $\hat{\psi}_i(k)$. We have

$$Limit_{ij}(k) = \min\{V/2, V\cos(\theta_{ij}(k))\}$$

Since, $\hat{\psi}_i(k)$ is in the direction of the bisector of angle $\psi_i(k)$, we have that for each $j \in N_i(k)$ the angle $\theta_{ij}(k) \le \psi_i(k)/2 < \pi/2$. Therefore, for a $\sigma \le V/2$ we have that the step size of agent i is bounded as follows:

$$\|p_i(k+1) - p_i(k)\| = \sigma\cos(\psi_i(k)/2) \le \frac{V}{2}\cos(\psi_i(k)/2) \le Limit_{ij}(k)$$

Hence, an agent i takes a step inside $AR_{ij}(k)$ for all $j \in N_i(k)$, and therefore, clearly, takes a step inside $AR_i(k)$.

$$p_i(k+1) = p_i(k) + \sigma\hat{\psi}_i(k)cos(\psi_i(k)/2) \subset AR_i(k)$$

Therefore, by Lemma 4 all agents of the system maintain visibility with their neighbours, as claimed in Corollary 1.

□

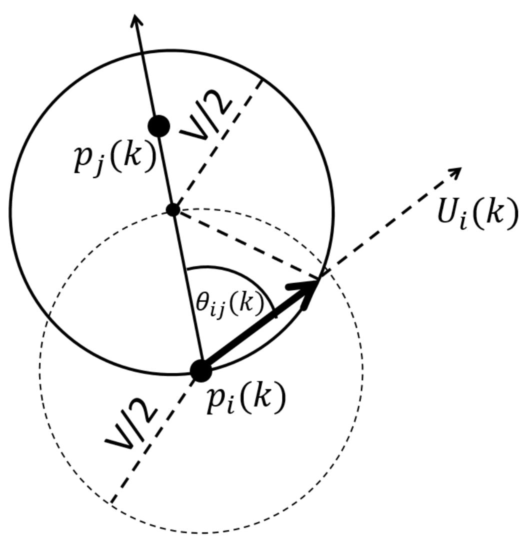

Figure 3: The section line of $AR_{ij}(k)$ along agent's i movement direction, $Limit_{ij}$, is marked by the wide arrow.

We have shown that the motion law (1) maintains the connectivity of the visibility graph. However, due to the agents' limited-visibility, the constellation of the agents may get stuck in cyclic sequence of time-steps, without gathering. We next give an example of such a situation.

Consider the constellation of agents presented in Figure 4. In this figure the agents 1, 2, 3 and 4, 5, 6 are located on parallel lines so that $\bar{p_1 p_3} \| \bar{p_4 p_6}$. These parallel lines are at a distance V from each other, and only $\bar{p_2 p_5}$ is perpendicular to $\bar{p_1 p_3}$ (and to $\bar{p_4 p_6}$), so that $\|p_2 - p_5\| = V$. Assume $\|p_1 - p_3\| = \|p_4 - p_6\| = \sigma < V$. Both p_1 and p_3 are not visible to p_4, p_5 and p_6 since they are distanced more than V from them, and both 4 and 6 are not visible to 1, 2 and 3. Considering the dynamic rule (1), where all agents are active at each time step, we have that at time-step k the wedge angles of 2 and 5 are $\psi_2(k) = \psi_5(k) = \pi$, therefore both 2 and 5 are locked. At time-step $k+1$ both agents 1 and 3 must move a step of size σ towards each other, so that they switch positions, and so do 4 and 6. The same switching phenomenon occurs over and over again simultaneously, leaving 2 and 5 locked forever, preventing the system from gathering.

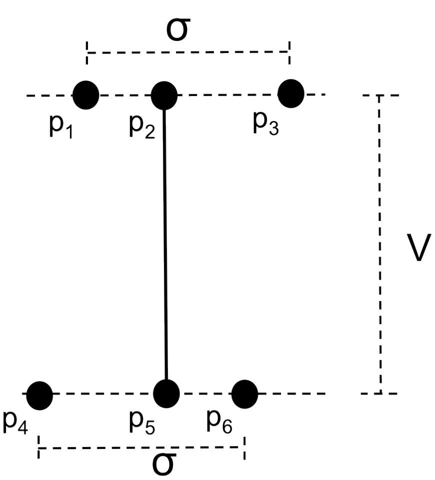

Figure 4: A special constellation that prevents the gathering of a multi-agent system of agents having a limited visibility acting by dynamic law (1).

This example shows that the deterministic schedule of motion (1) may lead to non-gathering constellations, hence some randomization is needed. Indeed adding randomization to the motion schedule breaks this "locked" situation and "free" the agents to move. For example, in the constellation above, if, once in a while, an agent "sleeps" and doesn't move (resulting, due to the jumps of 1 to 3 while 3 sleeps or due to the jump of 4 while 6 sleeps, in $\psi_2(k) = \pi/2$ or $\psi_5(k) = \pi/2$), agents 2 and 5 will approach each other, and eventually more agents will become visible to each other.

Gordon & Bruckstein in [1][2] also suggested a randomized rule of motion according to which each agent jumps a limited distance (σ) to a random location inside a region determined by the neighbours' directions, achieves gathering to a constellation within a disk of diameter equal to the visibility range (V), in a finite expected time. However, experimentally it was observed that, due to bearing only sensing, the agents' cluster was always of a smaller size, of the order of the step size σ, and was drifting in the plane in what seemed to be a random-walk.

We next modify the motion law discussed in [2], and prove that it gathers agents of the system to a disk with a radius equal to the agents' maximal step size σ within a finite expected number of time steps, Furthermore, as time tends to infinity the distribution of the agents' average position converges in probability to the distribution of a 2D random-walk.

Let us define $ar_i(k)$, a new allowable region of an agent i, which is contained in $AR_i(k)$, the allowable region given in [1]

$$ar_i(k) = D_{\frac{\sigma}{2}}\left(p_i(k) + \frac{\sigma}{2}U_i^-\right) \cap D_{\frac{\sigma}{2}}\left(p_i(k) + \frac{\sigma}{2}U_i^+\right) \tag{4}$$

where $\sigma < V/2$. See Figure 5. Recall that, if all agents take steps into their allowable regions ($AR_i(k)$), they all maintain visibility with their neighbours. Hence, the results concerning connectivity preservation apply to the new allowable regions ($ar_i(k)$).

We next show that, if the agents of the system jump to uniformly distributed random points in their "new" allowable regions, they will gather to a disk of radius σ. We have already proved gathering in a constellation starting with a complete visibility graph. But here, we first need to prove that, from an initial constellation corresponding to an abitrary but connected visibility graph, the system reaches a constellation with a complete visibility graph within a finite expected time, and subsequently remains that way. To show this, we follow [2][3], adjusting the timing model of the system to be semi-synchronised. Hence, the agents' assumed motion law is that, at any time-step k, each agent i has a strictly positive probability $\delta < 1$ to be active, and each active agent jumps to a uniformly distributed random point inside, its current allowable region, $ar_i(k)$.

$$p_i(k+1) = \begin{cases} p_i(k) & \text{if } \psi_i(k) \geq \pi \text{ or } \chi_i(k) = 0 \\ \\ \text{a random point in } ar_i(k) & \text{if } \psi_i(k) < \pi \text{ and } \chi_i(k) = 1 \end{cases}$$

$$\chi_i(k) = \begin{cases} 1 & \text{w.p. } \delta \\ 0 & \text{w.p. } 1-\delta \end{cases} \tag{5}$$

As mentioned above, we will prove that our system gathers to a disk of radius σ, within a finite expected number of time-steps. First, we shall show that any constellation having a connected visibility graph reaches a constellation with a complete visibility graph within a finite expected number of time-steps. Then, we show that, a constellation having a complete visibility graph, if the radius of the minimal enclosing circle of the system is greater than σ, it will significantly decrease within $\lceil n/2 \rceil$ time-steps with a strictly positive probability. Recall that, for $\sigma \leq V/2$, we have $ar_i(k) \subset AR_i(k)$. Therefore by Lemma 4, an agent can not lose sight

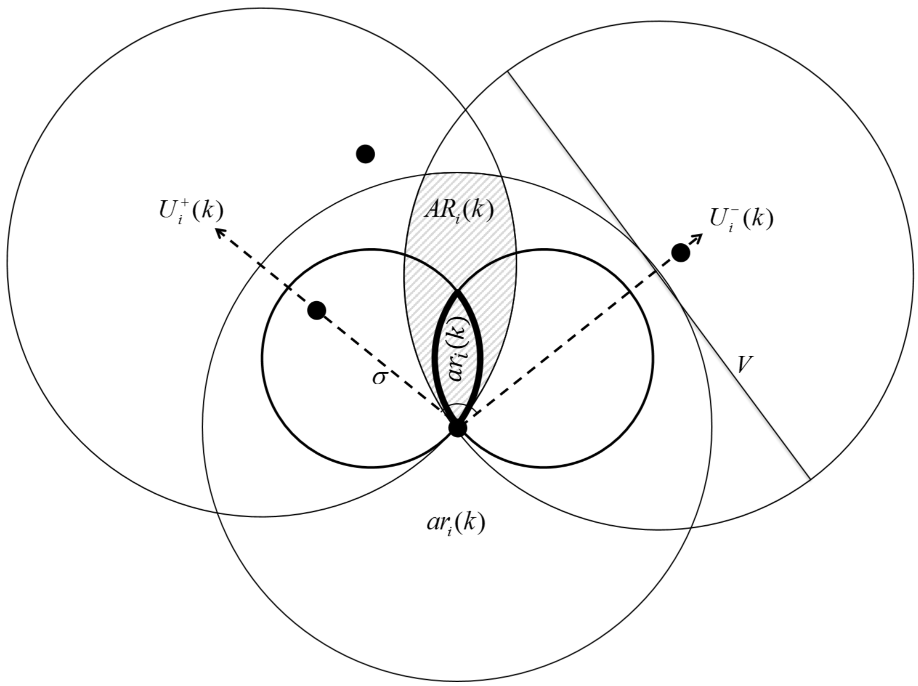

Figure 5: Allowable region according to (4): Note that, the dashed area created by the intersection of the circles of diameter V is a general allowable region $AR_i(k)$, and the thick line shape created by the intersection of the circles of diameter $\sigma < V/2$ is the new allowable region $ar_i(k)$.

of any of its neighbours, hence the connectivity of the visibility graph is preserved.

The outline of the first part of the proof is that at each time-step there is a strictly positive probability for a specific agent, located at the sharpest corner of the convex-hull, to be the only active agent, and to reduce its distance from $\bar{p}(k)$, the current average position of the agents, by a strictly positive quantity s^*. As a consequence the sum of all agents' squared distances of from $\bar{p}(k)$ is reduced by at least ${s^*}^2/n$ with a strictly positive probability $\delta(1-\delta)^{n-1}$. Hence, as long as the agents' interconnection graph is not complete, there is a bounded away from zero probability that it becomes complete within a finite number of time-steps. As a consequence of Lemma 4, once the agents visibility graph is complete it remains complete,. i.e. all the agents are henceforth confined to a disc of diameter V.

Let $CH(P(k))$ and $\partial CH(P(k))$ be the convex-hull of the agents' locations and the set of agents defining it (located at its corners). Let $\varphi_i(k)$ be the internal angle of the convex-hull's corner associated with an agent

$i \in \partial CH(P(k))$, and let $\mathcal{D}(P(k))$ be the diameter of the convex-hull, i.e.

$$\mathcal{D}(P(k)) \triangleq max_{i,j} \|p_i(k) - p_j(k)\|$$

***Lemma* 5.** *For any agent $i \in \partial CH(P(k))$, the distance between $p_i(k)$ and $\bar{p}(k)$ is bounded as follows:*

$$\|p_i(k) - \bar{p}(k)\| \geq \frac{\mathcal{D}(P(k))}{2n} \cos(\varphi_i(k)/2)$$

Proof. Any agent $i \in \partial CH(P(k))$, either defines the convex-hull diameter together with another agent j so that

$$\mathcal{D}(P(k)) = \|p_j(k) - p_i(k)\|$$

or there are two other agents j_1 and j_2 defining its diameter, so that

$$\mathcal{D}(P(k)) = \|p_{j_1}(k) - p_{j_2}(k)\|$$

By the general triangle inequality, we have that

$$max\{\|p_i(k) - p_{j_1}(k)\|, \|p_i(k) - p_{j_2}(k)\|\} \geq \frac{\mathcal{D}(P(k))}{2} \tag{6}$$

and we also have that:

$$\|\bar{p}(k) - p_i(k)\| = \|\frac{1}{n}\sum_j p_j(k) - p_i(k)\| = \|\frac{1}{n}\sum_j (p_j(k) - p_i(k))\|$$

Let $\theta_{ij}(k)$ be the angle between the vectors $p_j(k) - p_i(k)$ and U_i, a unit vector in the direction of the bisector of $\varphi_i(k)$.

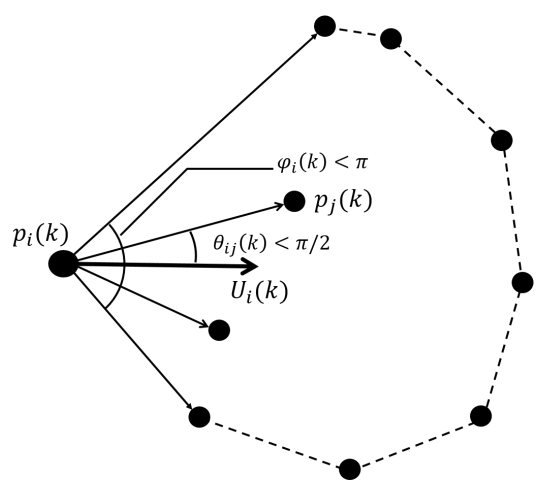

Figure 6: Since each angle of the convex-hull is smaller than π, any angle defined by an internal agent, a convex-hull corner and its associated bisector is smaller than $\pi/2$.

Since $\cos(\theta_{ij}(k)) > 0$ (see Figure 6), we have that

$$\|\bar{p}(k) - p_i(k)\| \geq \frac{1}{n}\sum_j \|p_j(k) - p_i(k)\| \cos\theta_{ij}(k)$$

Using (6), and that $\theta_{ij}(k) \leq \varphi_i(k)/2$ we have:

$$\|\bar{p}(k) - p_i(k)\| \geq \frac{1}{n}\sum_j \|p_j(k) - p_i(k)\| \cos\theta_{ij}(k) \geq \frac{\mathcal{D}(P(k))}{2n}\cos\left(\varphi_i(k)/2\right)$$

as claimed.

□

Using Lemma 5, let us show that if the diameter of the convex-hull is bounded away from zero, it has at least one corner farther from $\bar{p}(k)$ by a bounded away from zero value.

***Corollary* 2.** *If $\mathcal{D}(P(k))$ is bounded away from zero, the distance between $\bar{p}(k)$ and the position of s, the agent located at the sharpest corner of the system's convex-hull, is bounded away from zero as well.*

Proof. By Proposition 6, we have $\varphi_s(k) \leq \varphi_* = \pi(1-2/n)$, and by Lemma 5, we have that

$$\|p_s(k) - \bar{p}(k)\| \geq \frac{\mathcal{D}(P(k))}{2n}\cos(\varphi_s(k)/2)$$

Hence,

$$\|p_s(k) - \bar{p}(k)\| \geq \frac{\mathcal{D}(P(k))}{2n}\cos(\varphi_*/2)$$

i.e. the distance between $\bar{p}(k)$ and $p_s(k)$ is bounded away from zero as claimed.

□

***Lemma* 6.** *There exist strictly positive constants ρ^* and s^*, so that for any constellation $P(k)$, while $\mathcal{D}(P(k))$ is bounded away from zero, if agent s is active, the probability that at the next time-step it will be closer to $\bar{p}(k)$ by a distance greater than or equal to s^* is at least ρ^*.*

Proof. Let $\psi_s(k)$ be the angle of the minimal sector anchored at agent s's position and containing all its neighbours, so that $\psi_s(k) \leq \varphi_s(k) \leq \varphi_*$. Hence, by geometry, we have that the area of the allowable region of agent s is

$$\|ar_s(k)\| = \frac{1}{8}\sigma^2\left(\pi - \psi_s(k) - \sin(\psi_s(k))\right)$$

Angle $\psi_s(k)$ is upper bounded by φ_*, hence $\|ar_s(k)\|$ is bounded away from zero by a constant.

Let $D_{\|\bar{p}(k)-p_s(k)\|-s^*}(\bar{p}(k))$ be a disk centered at $\bar{p}(k)$ with the radius of $\|\bar{p}(k) - p_s(k)\| - s^*$, where s^* is a small but bounded away from zero value, so that if agent s jumps inside that disk, it is guaranteed to be closer to $\bar{p}(k)$ (comparing to where it was before the jump) by at least s^*.

The current agents' average position $\bar{p}(k)$ is located inside $CH(P(k))$, hence for any agent $i \neq s$ the angle $\angle p_i(k)p_s(k)\bar{p}(k)$ is smaller than or equal to $\varphi_s(k) \leq \varphi_*$. Furthermore, by Corollary 2, if the diameter of the system is bounded away from zero, the distance between $p_s(k)$ and $\bar{p}(k)$ is bounded away from zero, therefore for a small enough but significant s^*, the area of the intersection of $ar_s(k)$ and $D_{\|\bar{p}(k)-p_s(k)\|-s^*}(\bar{p}(k))$ is bounded away form zero as well (see Figure 7). Denote this intersection region by $F(k)$

$$F(k) \triangleq ar_s(k) \cap D_{\|\bar{p}(k)-p_s(k)\|-s^*}(\bar{p}(k))$$

Then, we have that the probability that agent s (if active) moves inside this region is strictly positive. We denote this probability by ρ^k, as follows:

$$\rho^k = \frac{\|F(k)\|}{\|ar_s(k)\|}$$

where $\|F(k)\|$ and $\|ar_s(k)\|$ are the areas of regions $F(k)$ and $ar_s(k)$, respectively, and argue that this value is strictly positive and bounded away from zero by ρ^*, a strictly positive constant, for all k, while $\mathcal{D}(k) > V$ (see Appendix 2). Hence, we have that whenever agent s is active and moves into area $F(k)$, it moves closer to $\bar{p}(k)$ by at least s^*. Therefore, the probability that agent s will be closer to $\bar{p}(k)$, is bounded away from zero by ρ_* as claimed.

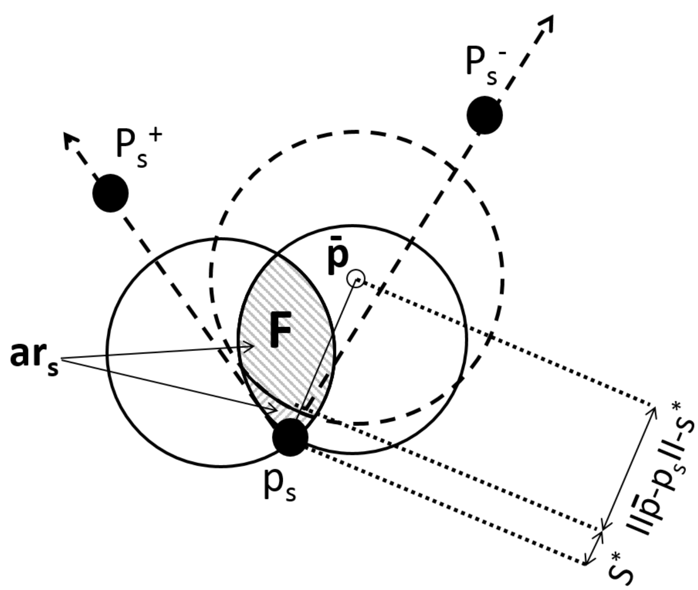

Figure 7: Given that agent s is active, the probability that the current distance of agent s from $\bar{p}$ will decrease in the next time-step is the proportion between the grey area F and the current allowable region ar_s of agent s.

□

Let $\mathcal{L}(P(k))$ be the sum of squared distances of all agents from their current average position, i.e.

$$\mathcal{L}(P(k)) = \sum_{i=1}^{n} \|p_i(k) - \bar{p}(k)\|^2$$

Lemma 7. *There is a probability of at least $\delta(1-\delta)^{n-1}\rho^*$ for a bounded away from zero $\mathcal{L}(P(k))$ to decrease by at least $\|s^*\|^2/n$ at each time-step.*

Proof. The probability that at some time-step k only the agent s becomes active is $\delta(1-\delta)^{n-1}$. The probability that agent s takes a step $\Delta p_s(k)$ of a size $\tilde{s} \geq s^*$ inside the region $F(k)$ is at least ρ^*, as shown in Lemma 6. In this case the value of $\mathcal{L}(P(k))$ decreases as follows:

$$\mathcal{L}(P(k+1)) - \mathcal{L}(P(k)) = \tilde{s}(\tilde{s} - 2\|p_s(k) - \bar{p}(k)\| \cos(\theta_s(k)) - \frac{\tilde{s}^2}{n} \qquad (7)$$

where $\theta_s(k) = \angle \bar{p}(k) p_s(k) p_s(k+1)$.

To prove this we proceed as follows:

$$\mathcal{L}(P(k+1)) = \sum_{i=1}^{n} \|p_i(k+1) - \bar{p}(k+1)\|^2 =$$

$$\sum_{\substack{i=1 \\ i\neq s}}^{n} \|p_i(k) - (\bar{p}(k) + \frac{\Delta p_s(k)}{n})\|^2 + \|p_s(k) + \Delta p_s(k) - (\bar{p}(k) + \frac{\Delta p_s(k)}{n})\|^2 =$$

$$\mathcal{L}(P(k)) + 2\frac{1}{n}\Delta p_s^\top(k)\left((n-1)(p_s(k) - \bar{p}(k)) - \sum_{\substack{i=1 \\ i\neq s}}^{n}(p_i(k) - \bar{p}(k))\right) +$$

$$+ \frac{(n-1) + (n-1)^2}{n^2}\|\Delta p_s(k)\|^2 =$$

$$\mathcal{L}(P(k)) + \|\Delta p_s(k)\|^2 + 2\Delta p_s^\top(k)(p_s(k) - \bar{p}(k)) - \frac{\|\Delta p_s(k)\|^2}{n} =$$

$$\mathcal{L}(P(k)) + \tilde{s}(\tilde{s} - 2\|p_s(k) - \bar{p}(k)\| \cos(\theta_s(k)) - \frac{\tilde{s}^2}{n}$$

By geometry, if agent s moves inside the disk $D_{\|\bar{p}(k) - p_s(k)\|}(\bar{p}(k))$, then we have

$$\tilde{s}(\tilde{s} - 2\|p_s(k) - \bar{p}(k)\| \cos(\theta_s(k)) < 0$$

hence,

$$\mathcal{L}(P(k+1)) - \mathcal{L}(P(k)) < -\frac{\tilde{s}^2}{n} < -\frac{s^{*2}}{n}$$

Therefore, as claimed There is a probability of at least $\delta(1-\delta)^{n-1}\rho^*$ that $\mathcal{L}(P(k))$ will decrease by at least $\frac{s^{*2}}{n}$.

□

Theorem 2. *For any initial constellation having a connected visibility graph, all agents gathers to a disk of diameter V in a finite expected number of time-steps.*

Proof. Since the initial agents' visibility graph is connected, $\mathcal{D}(P(k)) \leq (n-1)V$. Note that $\mathcal{D}(P)$ gets this maximal value when the agents are evenly distributed along a straight line, with a distance V between neighbours. Therefore $\mathcal{L}(P(k)) < n((n-1)V)^2$.

In addition, if $\mathcal{L}(P(k)) \leq (V/2)^2$ the agents' interconnection graph is necessarily complete, since the maximal distance of an agent from $\bar{p}(k)$ is $V/2$, and hence all inter-agent distances are necessarily less than V.

Therefore, the transition from any constellation comprising a connected visibility graph to a complete visibility graph constellation may be achieved by a finite number of possible steps M, where

$$M < \frac{n((n-1)V)^2 - V^2/4}{\|s^*\|^2/n} + 1$$

Let us examine the evolution of the agents' constellation every M steps. At the end of each series of M steps, the probability that $\mathcal{L}(P(k+M))$ will be less than $V^2/4$ is at least $(\delta(1-\delta)^{n-1}\rho^*)^M$. Therefore, by Proposition 7 the expected number of time-steps for gathering to a complete visibility graph is at most:

$$M\frac{1}{(\delta(1-\delta)^{n-1}\rho)^M}$$

By Lemma 4, once a complete visibility graph constellation is reached the system remains in such a constellation. Therefore, gathering to a disk of diameter V is achieved within a finite expected number of time-steps. □

From this point on, we shall prove that the agents further gather to a disk of radius σ, after having reached a constellation with complete visibility graph. We analyse the dynamics of the system considering the minimal enclosing circle of the agents locations, its radius and center being denoted by $R(k)$ and $C(k)$. We show that any agent is located at a distance greater than $\sigma/2$ from $C(k)$ can not jump to a greater distance from it, and if an agent located at a distance smaller than or equal to $\sigma/2$ from $C(k)$, it can not jump to a distance greater than σ from $C(k)$. Therefore, if $R(k) > \sigma$, it cannot increase. Furthermore, we show that there are at least two agents located on the circumference of the enclosing circle or within infinitesimal distances from it, that will most likely jump to positions closer to $C(k)$. Hence, if $R(k) > \sigma$, the radius of the smallest enclosing circle drops significantly within a batch of $\lceil n/2 \rceil$ time-steps with a strictly positive probability, and once $R(k)$ reaches σ it cannot outdistance it.

Without loss of generality, let $C(k)$ be at the origin of the $\mathbb{R}^2$-plane, and let $D_{\|p_i(k)\|}(0)$ and $D_\sigma(0)$ be disks centered at $C(k) = 0$ of radii $\|p_i(k)\|$ and σ.

***Lemma* 8.** *If all the agents of the system are within visibility range of each other, the allowable region of an agent i located at a distance greater than $\sigma/2$ from $C(k)$ is contained in $D_{\|p_i(k)\|}(0)$, and the allowable region of an agent located at a distance smaller than or equal to $\sigma/2$ from $C(k)$ is contained in $D_\sigma(0)$.*

Proof. Let us focus on the allowable regions of the agents located at the convex-hull's corners (since only they have non-zero allowable regions), and let us divide the current minimal enclosing disk into two half disks by a line defined by the points $p_i(k)$ and $C(P(k)) = 0$. Since all agents "see" each other, by Proposition 4, each one of the unit vectors $U_i^-(k)$ and $U_i^+(k)$ are necessarily pointing from $p_i(k)$ toward agents located at different half-disks (in some rare situations, where all agents lie on a line-segment with i located at its edge, both unit vectors are pointing towards the far end of the segment). Therefore, the intersection of the two disks of diameter σ defining i's allowable region, is necessarily contained in a disk of radius $\sigma/2$ centered at a distance $\sigma/2$ from $p_i(k)$ in the direction to the center of the enclosing disk (See figure 8), i.e.

$$ar_i(k) \subset D_{\frac{\sigma}{2}}(p_i(k) + \frac{C(k) - p_i(k)}{\|C(k) - p_i(k)\|})$$

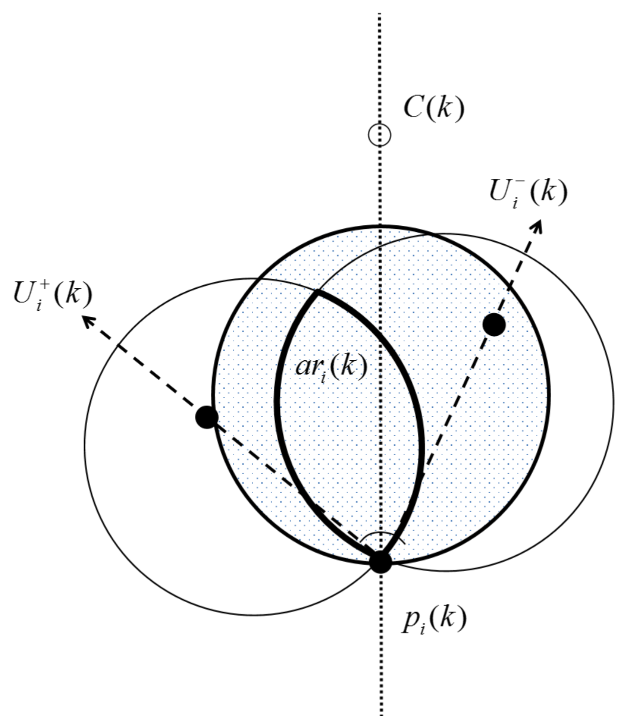

Figure 8: Allowable region of agent i (bordered by the thick line), contains in a disk of radius $\sigma/2$ centered at the distance $\sigma/2$ from $p_i(k)$ in the relative direction to $C(k)$ (the doted area), due to the fact that the two intersecting disks (bordered by the thin line circles), which comprises ar_i, are centered at both sides of the line defined by points $p_i(k)$ and $C(k)$ (dashed line).

Hence clearly, if $\|p_i(k)\| > \sigma/2$, agent i's allowable region is contained in the disk $D_{\|p_i(k)\|}(0)$, and if $\|p_i(k)\| \leq \sigma/2$, i's allowable region is contained in $D_\sigma(0)$, proving Lemma 8.

□

Lemma 9. *If all agents are within visibility range of each other, and $R(k)$ is greater than σ, then there exist strictly positive constants ρ^{**}, s^{**} and $\alpha < \sigma/2$, for any constellation P(k), so that at least two agents $s_\pm$ located within range of δ from the circumference of the smallest enclosing circle, have probabilities of at least ρ^{**} to jump to positions closer to $C(k)$ by distances of at least s^{**}.*

Proof. By Proposition 5, there exist two agents located on/or within a range of α from the circumference of the minimal enclosing circle and at corners of the convex-hull with inner-angles bounded away bellow π by a constant $(\varphi(\alpha, R(0)))$. We refer to any one of these two agents as agent s. Due to the fact that the angle $\psi_s(k)$ is upper bounded by $\varphi(\alpha, R(0)))$, the allowable region $ar_s(k)$ has an area bounded away from zero by a constant. We deal with $R(k) \geq \sigma$, and since $\alpha < \sigma/2$, we have that $\|p_s(k)\| > \sigma/2$. Then, by Lemma 8, the allowable region $ar_s(k)$ is contained in the disk $D_{\|p_s(k)\|}(C(k))$. Therefore, for a small enough but strictly positive constant s^{**}, we have that each one of the agents denoted by s has strictly positive probability to jump inside the disk $D_{R(k)-s^{**}}(0)$ (see Figure 9).

Furthermore, each one of these agents has a strictly positive probability to be active, δ, hence a strictly positive probability to jump inside the disk $D_{R(k)-s^{**}}(0)$, and as a consequence to significantly reduce its distance from $C(k)$. We denote this probability by $\rho_s(k)$, i.e.

$$\rho_s(k) = \frac{\|ar_{s^\pm} \cap D_{R(k)-s^{**}}(C(k))\|}{\|ar_s\|}$$

where operator $\|\cdot\|$ returns the area of the region $\cdot$, and argue that $\rho_s(k)$ is strictly positive and bounded away from zero by a constant ρ^{**}, in each time step k while $R(k) > \sigma$, where

$$\begin{aligned} \rho^{**} &= \frac{\pi\left(\frac{s^{**}}{2}\right)^2}{\frac{1}{2}\left(\frac{\sigma}{2}\right)^2(\pi - \varphi(\alpha, R_0) - \sin(\varphi(\alpha, R_0)))} \\ &\text{and} \\ s^{**} &= \frac{\sigma}{2}\left(1 - \sin\left(\frac{\varphi(\alpha, R_0)}{2}\right)\right) \end{aligned} \tag{8}$$

For the geometric derivation of these values, see Appendix 2. □

Theorem 3. *If all the agents are within visibility range of each other, they will gather to a disk of radius σ within a finite expected number of time-steps.*

Proof. By Lemma 5 the interior agents of disk $D_\sigma(0)$ can not jump out of it, and the exterior agents can not jump to distances farther from $C(k)$. Furthermore, by Lemma 9, there is a probability of at least $(\delta\rho^{**})^2$ that

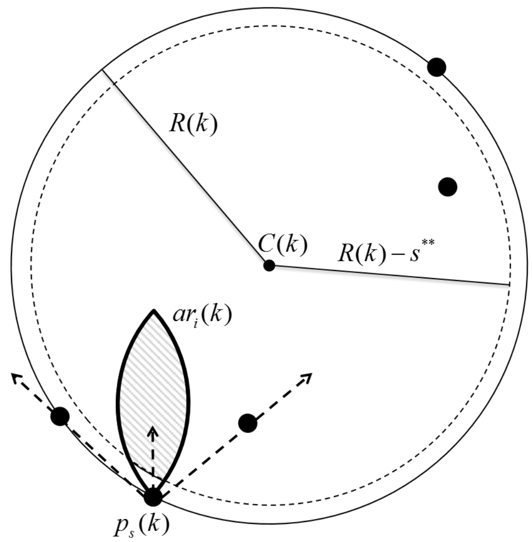

Figure 9: The allowable region of agent s have a significant area inside disk $D_{R(k)-s^{**}}(C(k))$, marked by the dashed area. Disks $D_{R(k)}(C(k))$ and $D_{R(k)-s^{**}}(C(k))$ are the interiors of the full-line circle and dashed circle respectively, and $ar_i(k)$ is the interior of the region bounded by the thick line.

at least two agents, located on the circumference of the minimal enclosing circle or within α distance from it, will jump to positions closer to $C(k)$ by at least s^{**}. By Proposition 7 of the Appendix, the expected number of time-steps for that event to occur is $(\delta\rho^{**})^{-2}$. Therefore, if the radius of the minimal enclosing circle is greater than σ, the expected number of time-steps for the radius of the minimal enclosing circle to decrease by at least s^{**} is at most $\lceil n/2 \rceil \lceil (\delta\rho^{**})^{-2} \rceil$, and consequently the expected number of time-steps for the radius of the minimal enclosing circle to decrease to σ is at most

$$\left\lceil \frac{R(k)-\sigma}{s^{**}} \right\rceil \lceil n/2 \rceil \lceil (\delta\rho^{**})^{-2} \rceil \leq \left\lceil \frac{V/2-\sigma}{s^{**}} \right\rceil \lceil n/2 \rceil \lceil (\delta\rho^{**})^{-2} \rceil$$

where s^{**} and ρ^{**} are given in (8). Once all agents are gathered to a disk of radius σ, by Lemma 8, they will remain confined to such a disk, hence the system gathers as claimed in Theorem 3.

□

5 Random dynamics analysis

We proved that the agents of the system gather to a disk of radius σ within a finite expected number of time steps. The simulations we performed showed that the "minimal confining disk" defined by the agents' constellation moves randomly in the plane.

We next prove that the centroid of the agents' constellation preforms a random motion, so that its location converges in probability to the distribution of a random-walk as k tends to infinity. To do so, we rely on a Theorem from [4] on the convergence in probability of random variables that are sums of uniformly bounded random increments. Given a random variable X_k so that

$$X_k = \sum_{i=1}^{k} Y_i, \text{ i.e. } Y_k = X_k - X_{k-1}$$

If Y_k satisfies

$$\mathbb{E}\left[Y_k | \mathcal{F}_{k-1}\right] = 0$$

where $\mathcal{F}_{k-1}$ is the sigma filed generated all prior realization of the process and

$$\sum_{i=1}^{k} Var\{Y_i\} \overset{a.s.}{\to} \infty$$

(i.e. the sum of variances tends to infinity with probability 1) then

$$\frac{X_{\tilde{k}}}{\sqrt{\nu}} \overset{p}{\to} N(0,1)$$

where $\tilde{k}$ is the stopping time as ν goes to infinity.

$$\tilde{k} = \min\{k : \sum_{k'=1}^{k} \mathbb{E}\{\|Y_{k'}\|^2\} > \nu\}$$

Let us analyse the long term behaviour of the random variable vectors $\bar{p}(k) = 1/n \sum_{i=1}^{n} p_i(k)$. Denote the step of agent i at time step k by $\Delta p_i = p_i(k+1) - p_i(k)$. Then, $\bar{p}(k)$ obeys

$$\bar{p}(k+1) = \frac{1}{n}\sum_{i=1}^{n} p_i(k+1) = \frac{1}{n}\sum_{i=1}^{n}(p_i(k) + \Delta p_i(k)) = \bar{p}(k) + \frac{1}{n}\sum_{i=1}^{n} \Delta p_i(k)$$

Therefore, we have to consider the sum of the jumps the agents make at each time-step.

Recall that, we have intentionally designed the motion law (5) so as to have that, in case the agents' constellation has a complete visibility graph, we ensure that $\mathbb{E}\{\bar{p}(k+1)|\bar{p}(k)\}$ is equal to $\bar{p}(k)$.

Let $\bar{ar}_i(k)$ be the mean position of the current allowable region of agent i. Then, if agent i is not located at a corner of the system's convex-hull, then $\psi_i(k) \geq \pi$, and therefore cannot jump, i.e. $p_i(k+1) = p_i(k)$. Otherwise, $\psi_i(k)$ is equal to the inner-angle of the convex-hull corner occupied by agent i, $\varphi_i(k) < \pi$. Then by (4), $\bar{ar}_i(k)$ is located at the center of $ar_i(k)$, as follows:

$$\bar{ar}_i(k) = \iint_{v \in ar_i(k)} v \mathrm{d}v = p_i(k) + \frac{\sigma}{2}\cos(\frac{\varphi_i(k)}{2})\hat{\psi}_i(k)$$

Since, an agent i located at a corner of the convex-hull stays put with probability $1-\delta$ and jumps with probability δ to a uniformly distributed random point in $ar_i(k)$, its expected position at the next time-step is

$$\mathbb{E}(p_i(k+1)) = p_i(k)(1-\delta) + \left(p_i(k) + \frac{\sigma}{2}\cos(\frac{\varphi_i(k)}{2})\hat{\psi}_i(k)\right)\delta$$

Hence, the expected agents' average position at the next time step is

$$\begin{aligned}
&\mathbb{E}(\bar{p}_i(k+1)) = \tfrac{1}{n}\sum_i \mathbb{E}(p_i(k+1)) = \\
&\tfrac{1}{n}\sum_{i\notin\partial CH(k)} p_i(k) + \tfrac{1}{n}\sum_{i\in\partial CH(k)}\Big(p_i(k) + \delta\tfrac{\sigma}{2}\cos(\tfrac{\varphi_i(k)}{2})\hat{\psi}_i(k)\Big) = \\
&\tfrac{1}{n}\sum_i p_i(k) + \delta\tfrac{\sigma}{2n}\sum_{i\in\partial CH(k)}\cos(\tfrac{\varphi_i(k)}{2})\hat{\psi}_i(k)
\end{aligned} \tag{9}$$

We have seen already in the proof of Lemma 1 that the sum of cosines of half the inner-angles of a convex-hull is zero.

$$\sum_{i\in\partial CH(k)}\cos(\frac{\varphi_i(k)}{2})\hat{\psi}_i(k) = 0 \tag{10}$$

hence we have that

$$\mathbb{E}\{\bar{p}(k+1)|p(k)\} = \bar{p}(k) \tag{11}$$

Let $\Delta\bar{p}(k)$, be the constellation centroid displacement at time step k, i.e $\Delta\bar{p}(k) = \bar{p}(k+1) - \bar{p}(k)$. Let S_k and X_k be the projections of the distribution of $\bar{p}(k)$ and $\Delta\bar{p}(k)$ on a unit vector with an arbitrary direction U. Then, by (11), we have that $\mathbb{E}\{\bar{X}(k+1)|X(k)\} = 0$.

Clearly, the increments $X(k)$ are uniformly bounded by $n\sigma$, and we next prove that the sum of their variances tends to infinity with probability 1. Let $Var(A)$ be the variance of a random variable A. Then, due to the fact that the random variables $\Delta p_i(k)$ and $\Delta p_j(k)$ are conditionally independent for $i \neq j$, we have that

$$Var(X_k|P(k)) = Var(U^\top\Delta\bar{p}(k)|P(k)) = Var(\frac{1}{n}U^\top\sum_i\Delta p_i(k) \geq \frac{1}{n^2}Var(U^\top\Delta p_s(k)|P(k))$$

where $P(k) = \{p_1(k), p_2(k), ..., p_1(k)\}$, and s is the agent located at the sharpest corner of the constellation's convex-hull.

The minimal value $Var(U^\top\Delta p_s(k))$ can assume is for a unit vector U orthogonal to $\hat{\psi}_s(k)$, i.e $U^\top\hat{\psi}_s(k) = 0$. Then, we have that:

$$Var(U^\top\Delta p_s(k)) = 2\delta^2\frac{\int\limits_{\frac{\sigma}{2}\sin\left(\frac{\psi_s(k)}{2}\right)}^{\frac{\sigma}{2}}(x - \frac{\sigma}{2}\sin\left(\frac{\psi_s(k)}{2}\right))^2\sqrt{\left(\frac{\sigma}{2}\right)^2 - x^2}dx}{\left(\frac{\sigma}{2}\right)^2\left(\frac{\pi-\psi_s(k)}{2} + \frac{sin(\psi_s(k))}{2}\right)} =$$

$$-2\delta^2\frac{\int\limits_0^{\frac{\pi-\psi_s(k)}{2}}\left(\frac{\sigma}{2}\right)^4\cos^3(\theta)\sin(\theta)d\theta}{\left(\frac{\sigma}{2}\right)^2\left(\frac{\pi-\psi_s(k)}{2} + \frac{sin(\psi_s(k))}{2}\right)} =$$

$$\delta^2\left(\frac{\sigma}{2}\right)^2\frac{1-\cos^4\left(\frac{\pi-\psi_s(k)}{2}\right)}{\frac{\pi-\psi_s(k)}{2} - \frac{1}{2}\sin(\pi - \psi_s(k))}$$

By Proposition 6, $\psi_s(k)$ is upper bounded by $\varphi_* = \pi(1-2/n)$, hence we have that

$$Var(U^\top \Delta p_s(k)) \geq \delta^2 \left(\frac{\sigma}{2}\right)^2 \frac{1-\cos^4\left(\frac{\pi-\varphi_*}{2}\right)}{\frac{\pi-\varphi_*}{2} - \frac{1}{2}\sin(\pi-\varphi_*)} = Var^*$$

Hence, $Var(X_k)$ is bounded away from zero by a strictly positive constant Var^*, and therefore the infinite sum of the X_k's variances tends to infinity almost surely.

Hence, by Theorem 35.11 in [4], we have that

$$\frac{S_\nu}{\sqrt{\mathcal{V}}} \xrightarrow{p} N(0,1)$$

where ν is the stopping time as $\mathcal{V}$ goes to infinity

$$\nu = \min\{t : \sum_{k=1}^{t} Var(X_k) > \mathcal{V}\}$$

Assume the mean value of the variances of the increments X_k converges in probability to η^2, i.e

$$\sum_k Var(X_k) \xrightarrow{p} k\eta^2$$

Then, we have that

$$\frac{S_k}{\sqrt{k}\eta} \xrightarrow{p} N(0,1)$$

i.e the distribution of S_k converges in probability to the distribution of random-walk with steps of the size η.

Hence, we have that the projection of the random vector $\bar{p}(k)$ on an arbitrary (constant) direction U converges to a normal distribution with a variance $k\eta^2$.

We ran multiple simulations, and used the results to estimate η. We show part of the simulation results in Figure 10, and the analysis of η in Figure 11. Interestingly, the average random walk step size, η, is inversely dependent on the numbers of agents

$$\eta \propto \frac{1}{n}$$

6 Discussion

We showed that a system of identical, anonymous and oblivious agents having limited visibility and bearing only sensing starting with an initial constellation with a connected visibility graph gathers to a disk with radius equal to the agents' maximal allowed step size within a finite expected number of time steps. Furthermore, we proved that, after the visibility graph of the constellation becomes complete, the expected location of the centroid of the constellation remains in place, while the distribution of the random centroid location converges to that of a random-walk dynamics.

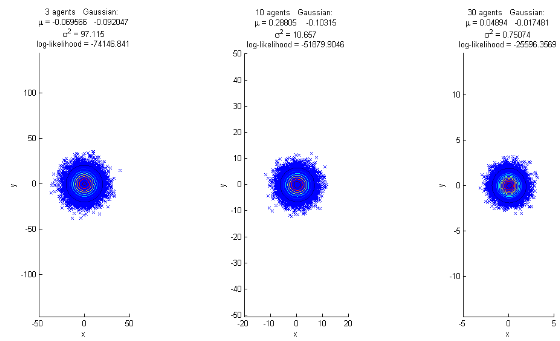

Figure 10: Gaussian fit for the average position of the systems' constellations at time-step 1000. This results are for 10000 simulations with random initial constellations of $\{3, 10, 30\}$ agents.

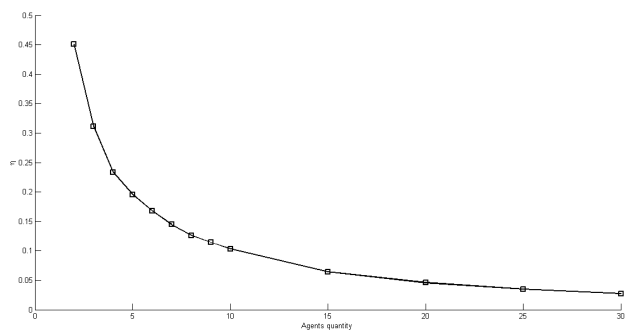

Figure 11: Estimation of η from Simulations results. Analysis of the results of 10000 simulations with random initial constellations of $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 25, 30\}$ agents. We preformed Gaussian fit for the average position at time-step 1000, for all the results having the same number of agents.

We may generalize the motion law to one preforming convergence in distribution to a random-walk dynamics, as follows: Each agent i jumps to a random point with a general, non-uniform 2D distribution function bounded inside $AR_i(k)$, the original allowable region presented in [1][2], with average location centroid at point $p_i(k) + \alpha\hat{\psi}_i(k)\cos(\psi_i(k)/2)$, where α is a strictly positive constant (recall that

$$AR_i(k) \triangleq \left(\bigcap_{j \in N_i(k)} D_{\frac{V}{2}}(c_{ij}(k)) \right) \cap D_{\frac{V}{2}}(p_i(k))$$

and $c_{ij}(k) = p_i(k) + V/2(p_j(k) - p_i(k))/\|p_j(k) - p_i(k)\|$).

In the future we intend to study a control mechanism applied on this system, which we call broadcast control. This control model assumes that, we work with similar, non-communicating agents which have the ability to "hear" a control broadcast with strictly positive probability at each time step. The control broadcast is a vector directing the agent that hears it towards a specific direction, for example by adding a "virtual" agent in the control direction.

Figure 12 described a geometrical relations between allowable regions of different agents in a constellation having a complete visibility graph. As can be seen in (c), there will be at least one agent whose allowable region can be affected from the received broadcast, cancelling the ability of the agent to jump in a direction opposite to control vector. Therefore the expected agents' average position will not remain at the current average position, but rather will drift in the desired direction.

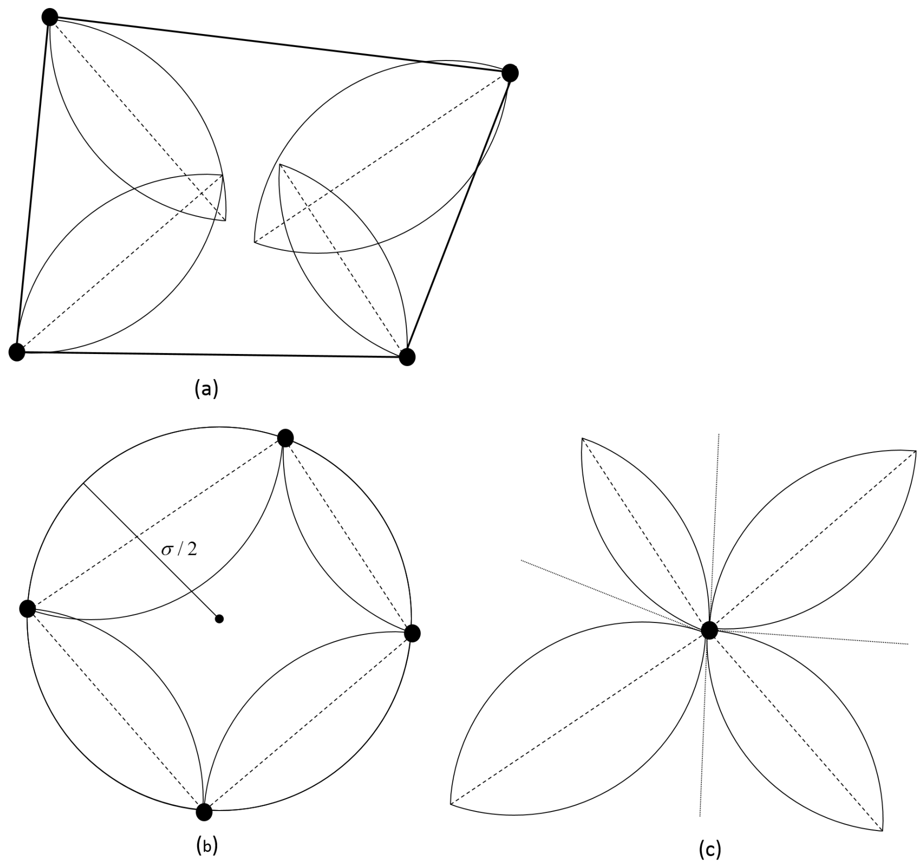

Figure 12: All allowable regions. (a) Allowable regions localization. (b) Allowable regions right boundaries forms a circle of radius $\sigma/2$. (c) Allowable regions of agents located at following corners tangent to each other. The doted lines are the tangent border lines.

Appendix 1 - Geometry and Probability Results

Proposition 4. *Let $O(P)$ be the minimal enclosing circle of P a set of points in $\mathbb{R}^2$. Then, any partition of $O(P)$ into two by a line passing through its center results in half circles with at least one point of P on each.*

Proof. Let $R(P)$ and $C(P)$ be the radius and center of $O(P)$ respectively. We have that

$$R(P) \triangleq \min_{C(P)\in\mathbb{R}^2} \left\{ \max_{p_i\in P} \|p_i - C(P)\| \right\}$$

and therefore for any point $p_j \in P$ which not located on the perimeter of $O(P)$ we have that $O(P) = O(P \setminus p_j)$. Hence,

$$O(P) = O(\partial P) \tag{12}$$

where $\partial P \subseteq P$ is the subset of agents lying on $O(P)$'s perimeter.

Assume we may cut $O(P)$ into two equal arcs by a line crossing point $C(P)$ which orthogonal to a unit vector U, where none of the points from P lie on one of those arcs. Let $p \in \partial P$ be the closest point to this line, and let d be the distance between p and the line (see Figure 13). Then, all the points of set ∂P are contained in $O'(\partial P)$, a circle of the radius $\sqrt{R^2 - d^2}$ centred at $C' = C + dU$, so that $O(\partial P)$ is smaller than $O(P)$, which contradicts (12). Hence, the assumption cannot be true, and we have that there must be at least one point on each half circle mentioned above, proving Proposition 4.

□

Proposition 5. *Given a constellation with a finite number of points $n > 1$ in the $\mathbb{R}^2$-plane, and a strictly positive diameter D. There are at least two points within a distance $\alpha > 0$ from the circumference of the constellation's smallest enclosing circle, at corners of the constellation's convex-hull with inner angles bounded away bellow π by a strictly positive value (dependent on α).*

Proof. There are at least two corners of the constellation's convex-hull located on the circumference of the minimal enclosing circle. Let us denote the points located at these corners by p. Let c be the center of the enclosing circle, let l be a line passing trough the point $p + \alpha(c-p)/\|c-p\|$ and perpendicular to the vector $c - p$. Let F be the region created by the intersection of the smallest enclosing circle and the half plane bounded by l which includes point p, and denote the set of the constellation's points located in F by P_α (see Figure 14).

Assuming the number of point in P_α is $m \leq n-1$, then if we cut the constellation's convex-hull with l into a two convex polygons, the sum of the inner-angles of the polygon which includes point p is upper bounded by $\pi(m+2-2)$, since this polygon is comprised by the points of the set P_α and the two points created by the intersection with l, denoted by $\pm$. By geometry, the inner-angles of the above mentioned polygon corners at the

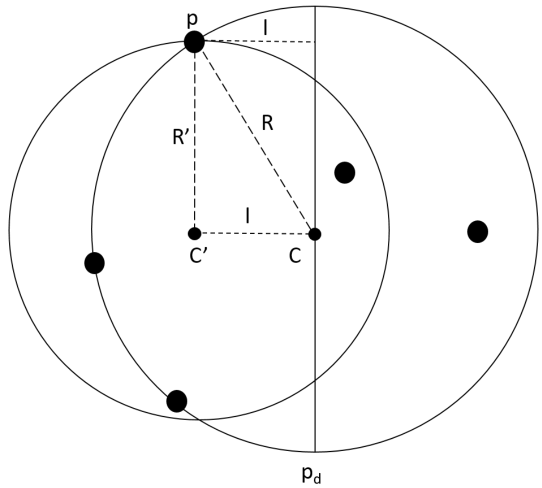

Figure 13: Orientation figure for the proof of proposition 4. The full line dividing the the minimal enclosing circle (full line circle) into two arcs with equal length. p is the point closet to the dividing line from the set of points lying on the minimal enclosing circle, which by assumption are all located only on one arc of the enclosing circle, so that all of the points of this set are located inside the smaller and dashed circle.

points ± are lower bounded by $\beta = atan\left(\alpha/\sqrt{R^2 - \alpha^2}\right)$ (see Figure 14). Hence, the average value of the inner-angles associated with the points of the set P_α is upper bounded by

$$\varphi(R,\alpha) = \frac{\pi(m+2-2) - 2\beta}{m} = \pi - \frac{2\beta}{m}$$

and therefore the inner-angle of the sharpest corner of the convex-hull of the set P_δ is upper bounded as well by

$$\varphi(R,\alpha) = \pi - \frac{2atan\left(\alpha/\sqrt{R^2-\alpha^2}\right)}{m}$$

Since at least two corners of the constellation's convex-hull are located on the circumference of the minimal enclosing circle, we have that, there are at least two corners of the convex-hull located within a strictly positive distance of $\alpha < R$ from the circumference of the minimal enclosing circle with angles bounded away form π by a value equal to $\varphi(R,\alpha)$.

□

Proposition 6. *The sharpest corner of the convex-hull of any n points in $\mathbb{R}^2$ is upper bounded by $\varphi_* = \pi(1 - 2/n)$.*

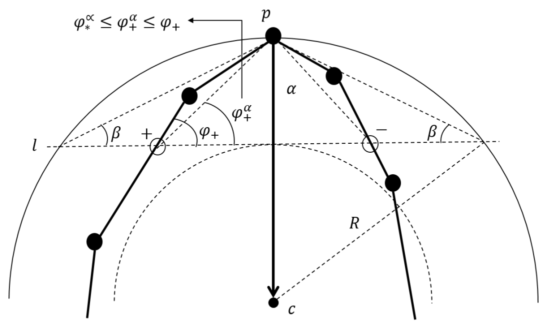

Figure 14: Upper bound of the two angles of corners of the constellation's convex-hull which located within a strictly positive distance $\alpha < R$ from the circumference of the constellation's minimal enclosing circle. Point p is located on the circumference of the minimal enclosing circle centred at point c, and the angles $\varphi_\pm$ are lower bounded by β, as may be seen in the figure.

Proof. For any convex polygon with $m \le n$ corners, the sum of the corners' inner angles is $\pi(m-2)$, and the average inner-angle is $\pi(1-\frac{2}{m})$. Therefore, φ_s the interior angle of the polygon's sharpest corner is necessarily smaller than or equal to $\pi(1-\frac{2}{m})$. Since, we deal with the convex-hull of n points, we have that

$$\varphi_* = \pi(1-2/n) \ge \pi(1-2/m) \ge \varphi_s$$

□

Proposition 7. *Assume that, at each time-step an event occurs with probability $p < 1$, then the expected number of time-steps for the first event to occur is p^{-1}.*

Proof. The probability that the first event occurs at exactly time-step k is $(1-p)^{k-1}p$. Therefore, the expected number of time-steps for the first event to occur is

$$\sum_{k=1}^{\infty} k(1-p)^{k-1}p = -p\frac{d}{dp}\sum_{k=1}^{\infty}(1-p)^k = -p\frac{d}{dp}\left(\sum_{k=0}^{\infty}(1-p)^k - 1\right) = -p\frac{d}{dp}\frac{1}{p} = \frac{1}{p}$$

□

Appendix 2 - Bounds of constants

In the following section we derive the lower bounds of the probabilities (ρ^*/ρ^{**}) that the agents marked by s in Lemmas 6 and 9 will jump to locations closer to $\bar{p}(k)$ or $C(k)$ by bounded away form zero values (s^*/s^{**}).

In order to simplify our calculations, we define the following quantities related to the allowable regions of agent s. Let $\|ar_s(k)\| = \sigma^2/2(\pi - \psi_s(k) - \sin(\psi_s(k))/2)$ be the area of agent s's allowable region. Note that $\|ar_s(k)\|$ is upper bounded by $\|ar_s\|_{max} = \pi\sigma^2/4$. Let $p_s^h(k) = p_s(k) +$

$\hat{\psi}_s(k)\sigma\cos(\psi_s(k)/2)$ the meeting point of the two arcs bounding $ar_s(k)$ which is not located at $p_s(k)$, and let $h(\psi_s(k)) = \sigma\cos(\psi_s(k)/2)$ be length of the line segment $[p_s(k), p_s^h(k)]$. Let $v(\psi_s(k)) = \sigma(1 - \sin(\psi_s(k)/2))$ be the distance between these two arcs' middle points.

The bounds s^* and ρ^* in Lemma 6

Without loss of generality, let $\bar{p}(k)$ be the origin of the $\mathbb{R}^2$-plane.

By Proposition 6, the angle of the sharpest corner of the convex-hull, occupied by agent s, is bounded by $\varphi_* = \pi(1 - 2/n)$. Being $\bar{p}(k)$ inside the convex-hull, we have that $\hat{\psi}_s(k)$, the bisector of the angle $\psi_s(k)$, is pointing from $p_s(k)$ inside the disk $D_{\|p_s(k)-\bar{p}(k)\|}(\bar{p}(k))$. Denote the angle between $\hat{\psi}_s(k)$ and the vector pointing from $p_s(k)$ to $\bar{p}(k)$ by θ. Note that, since $\bar{p}(k)$ is located inside the constellation's convex-hull θ is upper bounded by $\varphi_*/2$.

Our calculations address two cases, in the first point $p_s^h(k)$ is inside the disk $D_{\|p_s(k)\|}(\bar{p}(k))$, and in the second case this point is not in the disk $D_{\|p_s(k)\|}(\bar{p}(k))$.

If $p_s^h(k)$ is inside the disk $D_{\|p_s(k)\|}(\bar{p}(k))$, we have that the distance between $\bar{ar}_s(k)$, the centroid of $ar_s(k)$, and $\bar{p}(k)$ is as follows:

$$\|\bar{ar}_s(k)\| = \sqrt{\|p_s(k)\|^2 + \left(\frac{h(\psi_s(k))}{2}\right)^2 - 2\|p_s(k)\|\frac{h(\psi_s(k))}{2}\cos(\theta)} \leq$$

$$\sqrt{\|p_s(k)\|^2 + \left(\frac{h(\varphi_*)}{2}\right)^2 - 2\|p_s(k)\|\frac{h(\varphi_*)}{2}\cos\left(\frac{\varphi_*}{2}\right)} =$$

$$\sqrt{\|p_s(k)\|^2 + \frac{h(\varphi_*)}{2}\left(\frac{h(\varphi_*)}{2} - 2\|p_s(k)\|\cos\left(\frac{\varphi_*}{2}\right)\right)}$$

Since $p_s^h(k)$ is inside the disk $D_{\|p_s(k)\|}(\bar{p}(k))$, we have that $h(\varphi_*) \leq 2\|p_s(k)\|\cos(\varphi_*/2)$ (i.e. $\sigma \leq 2\|p_s(k)\|$), and therefore

$$\|\bar{ar}_s(k)\| \leq \sqrt{\|p_s(k)\|^2 - \left(\frac{h(\varphi_*)}{2}\right)^2} = \|\bar{ar}_s(k)\|_{max}$$

Choosing $s^k = \|p_s(k)\| - \|\bar{ar}_s(k)\|_{max}$ as the lower bound of agent s's distance reduce from $\bar{p}(k)$ after taking a step, we have that there is at least half disk of diameter $v(\psi_s(k))/2 \geq v(\varphi_*)/2$ inside the allowable region $ar_s(k)$ and inside the disk $D_{\|p_s(k)\|-s^k}(\bar{p}(k))$, as may be seen in Figure 15.

By Corollary 2, for an agent's constellation having a diameter greater than or equal to V, we have that

$$\|p_s(k)\| \geq \frac{V}{2n}\cos\left(\frac{\varphi_*}{2}\right) = \frac{V}{2n}\sin\left(\frac{\pi}{n}\right)$$

and therefore

$$s^k = \|p_s(k)\| - \sqrt{\|p_s(k)\|^2 - \left(\frac{h(\varphi_*)}{2}\right)^2} = \|p_s(k)\|\left(1 - \sqrt{1 - \left(\frac{h(\varphi_*)}{2\|p_s(k)\|}\right)^2}\right) \geq$$

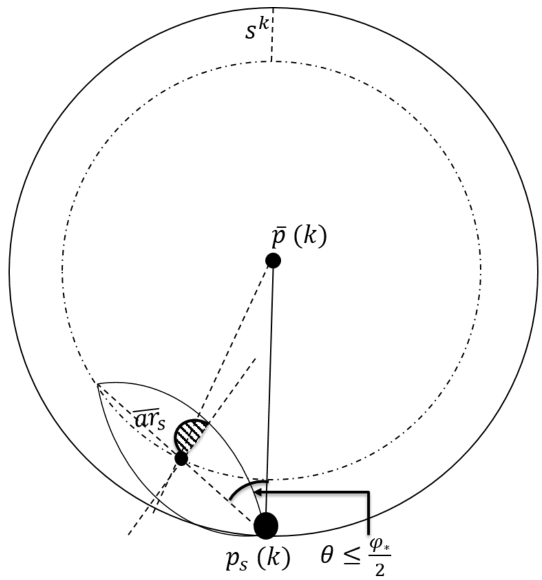

Figure 15: The bounds s^* and ρ^*: if $p_s^h(k)$ is inside the disk $D_{\|p_s(k)\|}(\bar{p}(k))$, then we have that at least the half disk, marked by the dashed area, is inside the allowable region $ar_s(k)$ and inside the disk $D_{\|p_s(k)\|-s^k}(\bar{p}(k))$.

$$\|p_s(k)\|\left(1-\sqrt{1-\left(\cos\left(\frac{\varphi_*}{2}\right)\right)^2}\right) \geq \frac{V}{2n}\sin\left(\frac{\pi}{n}\right)\left(1-\sqrt{1-\left(\sin\left(\frac{\pi}{n}\right)\right)^2}\right) = s^*$$

Hence, if active, the probability of agent s to reduce its distance from $\bar{p}(k)$ by at least s^k is lower bounded by

$$\rho^k = \frac{\frac{1}{2}\pi\left(\frac{v(\psi_s(k))}{2}\right)^2}{\|ar_s(k)\|} \geq \frac{\frac{1}{2}\pi\left(\frac{v(\varphi_*)}{2}\right)^2}{\|ar_s(k)\|_{max}} = \frac{\frac{1}{2}\pi\left(\frac{\sigma}{2}\right)^2\left(1-\sin\left(\frac{\varphi_*}{2}\right)\right)^2}{\pi\left(\frac{\sigma}{2}\right)^2} =$$

$$\frac{1}{2}\left(1-\cos\left(\frac{\pi}{n}\right)\right)^2 = \rho^*$$

If point $p_s^h(k)$ is outside the disk $D_{\|p_s(k)\|}(\bar{p}(k))$, we have that a line-segment through the point $\bar{p}(k)$ and orthogonal to $\hat{\psi}_s(k)$, crossing the segment $[p_s(k), p_s^h(k)]$ at a distance $\|p_s(k)\|\cos(\theta) \geq \|p_s(k)\|\cos(\varphi_*/2)$ form point $p_s(k)$ (see Figure 16).

Note that, the part of the above mentioned line segment which located inside $ar_s(k)$ has a length of at least

$$\|p_s(k)\|\cos\left(\frac{\psi_s(k)}{2}\right)\frac{v(\psi_s(k))}{h(\psi_s(k))} \geq \frac{V}{2n}\sin^2\left(\frac{\pi}{n}\right)\frac{v(\psi_s(k))}{h(\psi_s(k))} \geq$$

$$\frac{V}{2n}\sin^2\left(\frac{\pi}{n}\right)\left(1-\cos\left(\frac{\pi}{n}\right)\right) = d^*$$

(see Figure 16). Hence, if we choose s^k as follows:

$$s^k = \|p_s(k)\| \left(1 - \sin\left(\frac{\varphi_*}{2}\right)\right)$$

by corollary 2, we have that

$$s^k \geq \frac{V}{2n} \sin\left(\frac{\pi}{n}\right)\left(1 - \cos\left(\frac{\pi}{n}\right)\right) = s^*$$

and as a consequence there is at least half disk of diameter d^* inside the allowable region $ar_s(k)$ and inside the disk $D_{\|p_s(k)\|-s^*}(\bar{p}(k))$, so that

$$\rho^k \geq \frac{\frac{1}{2}\pi\left(\frac{d^*}{2}\right)^2}{\|ar_s(k)\|} \geq \frac{\frac{1}{2}\pi\left(\frac{\frac{V}{2n}\sin^2\left(\frac{\pi}{n}\right)\left(1-\cos\left(\frac{\pi}{n}\right)\right)}{2}\right)^2}{\pi\left(\frac{\sigma}{2}\right)^2} = \rho^*$$

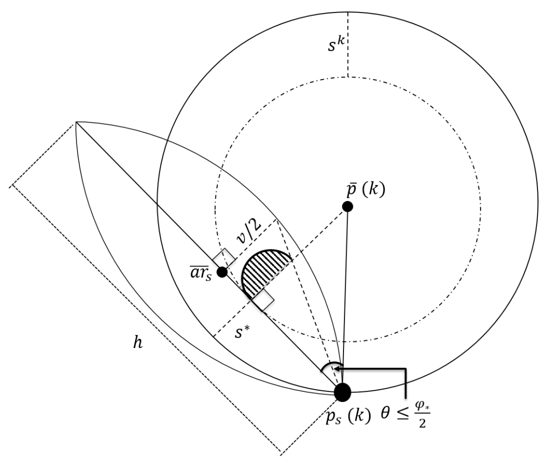

Figure 16: The bounds s^* and ρ^*: if $p_s^h(k)$ is located outside the disk $D_{\|p_s(k)\|}(\bar{p}(k))$, then we have that at least the half disk, marked by the dashed area, is inside the allowable region $ar_s(k)$ and inside the disk $D_{\|p_s(k)\|-s^*}(\bar{p}(k))$.

Therefore, in both cases the quantities s^k and ρ^k are bounded by a strictly positive constants s^* and ρ^* independent of k.

The bounds s^{**} and ρ^{**} in Lemma 9

We use a result of Lemma 8, which state that if an agent s is located with a distance greater then or equal to $\sigma/2$ from $C(k)$, the center of the minimal enclosing circle of the current agent-constellation, and it's allowable region, $ar_s(k)$, is contained inside the disk $D_{\|p_s(k)-C(k)\|}(C(k))$.

Without loss of generality, we let point $C(k)$ to be the origin of the $\mathbb{R}^2$-plane. Let θ be the angle between the vectors $-p_s(k)$ and $\hat{\psi}_s(k)$. By Proposition 5, we have that the angle $\psi_s(k)$ is upper bounded by $\varphi(R_0,\alpha) < \pi$, hence by Proposition 4, we have that $\theta \leq \varphi(R_0,\alpha)/2$.

The distance between points $C(k)$ and $\bar{a}r_s(k)$ is

$$\|\bar{a}r_s(k)\| = \sqrt{\|p_s(k)\|^2 + \left(\frac{h(\psi_s(k))}{2}\right)^2 - 2\|p_s(k)\|\frac{h(\psi_s(k))}{2}\cos(\theta)}$$

hence choosing s^k as $\|p_s(k)\| - \|\bar{a}r_s(k)\|$ we have that the allowable region $ar_s(k)$ includes at least half disk of diameter $v(\psi_s(k))/2$, half the distance between the two middle points of the arcs defining $ar_s(k)$ (see figure 17).

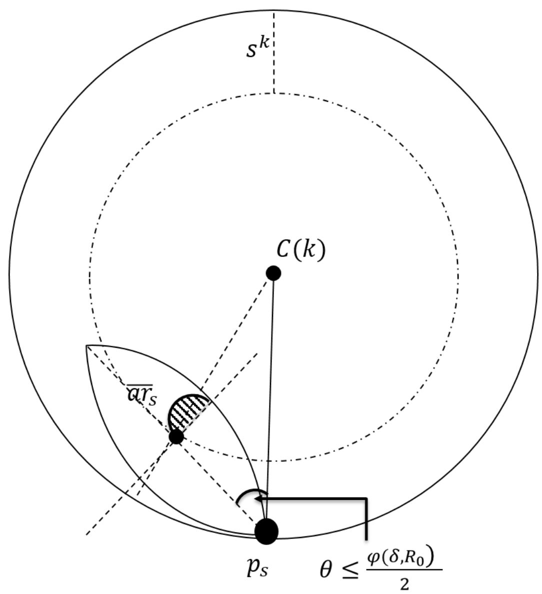

Figure 17: The bounds s^{**} and ρ^{**}: choosing s^k as the $\|p_s(k) - C(k)\| - \|\bar{a}r_s(k) - C(k)\|$ results with at least the half disk, marked by the dashed area, inside the allowable region $ar_s(k)$ and inside the disk $D_{\|p_s(k)\|-s^k}(\bar{p}(k))$.

Since, we deal with an agent located at a distance greater then $V/2$ from $C(k)$, and $\theta \leq \varphi(R_0, \alpha)/2$, we have that

$$s^k \geq \frac{V}{2} - \sqrt{\left(\frac{V}{2}\right)^2 + \left(\frac{\sigma}{2}\cos\left(\frac{\varphi(R_0,\alpha)}{2}\right)\right)^2 - 2\left(\frac{V}{2}\right)\frac{\sigma}{2}\cos^2\left(\frac{\varphi(R_0,\alpha)}{2}\right)} = s^{**}$$

and

$$\rho^k \geq \frac{\frac{1}{2}\pi\left(\frac{\sigma}{4}\left(1 - \sin\left(\frac{\varphi(R_0,\alpha))}{2}\right)\right)\right)^2}{\|ar_s(k)\|} \geq \frac{\frac{1}{2}\pi\left(\frac{\sigma}{4}\left(1 - \sin\left(\frac{\varphi(R_0,\alpha)}{2}\right)\right)\right)^2}{\pi\left(\frac{\sigma}{2}\right)^2} =$$

$$\frac{1}{8}\left(1 - \sin\left(\frac{\varphi(R_0,\alpha)}{2}\right)\right)^2 = \rho^{**}$$

Therefore, the quantities s^k and ρ^k are bounded by a strictly positive constants s^{**} and ρ^{**} independent of k.

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