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Continuous time gathering of agents with limited visibility and bearing-only sensing

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Abstract A group of mobile agents, identical, anonymous, and oblivious (memoryless), able to sense only the direction (bearing) to neighboring agents within a finite visibility range, are shown to gather to a meeting point, in finite time, by applying a very simple rule of motion. The agents act in continuous time, and their rule of motion is as follows: they determine the smallest visibility disk sector in which all their visible neighbors reside. If this disk sector spans an angle smaller than π, then they set the velocity vector to be the sum of the two unit vectors in $\mathbb{R}^2$ pointing to the extremal neighbors. Otherwise, they do not move. If the initial constellation of agents has a visibility graph that is connected, we prove that the agents gather to a common meeting point in finite time.

Keywords Finite time gathering · Bearing-only sensing · Agents with finite visibility sensing

1 Introduction

This paper studies the problem of mobile agent convergence, or robot gathering, under severe limitations on the capabilities of the agents. We assume that the agents are point-like and move in the environment (the plane $\mathbb{R}^2$) according to what they currently sense in their neighborhood. All agents are identical and indistinguishable (that is, they are anonymous and have no identifiers), and all of them perform the same "reactive" rule of motion in response to what they sense. Our assumption will be that the agents have a finite visibility range V, a distance beyond which they cannot sense the presence of other agents. The agents within the visibility disk determined by the sensing range V around each agent are defined as its

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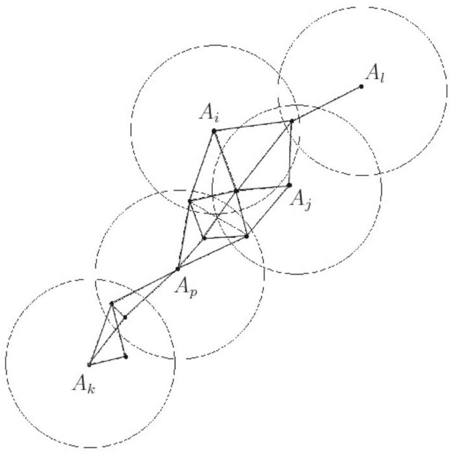


Fig. 1 A constellation of agents in the plane displaying the visibility disks of agents A_k, A_l, A_i, A_j, A_p and the visibility graph that they define, having edges connecting pairs of agents that can see each other

neighbors. We further assume that each agent can only sense the direction to its neighbors, i.e., it performs bearing-only measurements yielding unit vectors pointing toward all its neighbors. In our setting, each agent continuously senses its neighbors within the visibility disk and sets its motion only according to the distribution of unit vectors pointing to its current neighbors. Figure 1 shows a constellation of agents in the plane ($\mathbb{R}^2$), their visibility graph and the visibility disks of some of them. Each agent moves based on the set of unit vectors pointing to its neighbors.

In this paper we shall prove that continuous time limited visibility sensing of directions only and continuous adjustment of agents' velocities according to what they sense is enough to ensure the gathering of the agents to a point of encounter in finite time.

The literature of robotic gathering is vast, and the problem was addressed under various assumptions on the sensing and motion capabilities of the agents, see for example the recent survey by Barel et al. (2016). Here we shall mostly mention papers that deal with gathering assuming continuous time motion and limited visibility sensing, since these are most relevant to our work. Olfati-Saber et al. (2007) nicely survey the work on the topic of gathering (also called consensus) for networked multi-agent systems, where the connections between agents are not necessarily determined by their relative position or distance. This approach to multi-agent systems was indeed the subject of much investigation, and some of the results, also involving switching connection topologies, are useful in dealing with dynamics based on visibility-defined neighbor constellations too. A lot of work was invested in the analysis of multi-agent system dynamics, where agents sense each other through distance-based influence fields. A prime example is the very influential work of Gazi and Passino (2006) which analyses the stability of a clustering process. Interactions involving hard limits on the visibility distance in sensing neighbors were analyzed in relatively few works. Ji and Egerstedt (2007) analyzed such problems using potential functions that are distance-based barrier functions setting the visibility range. They proved connectedness-preservation properties at the expense of making some agents temporarily identifiable and traceable via a hysteresis process. Ando et al. (1999) were the first to deal with hard constraints of limited visibility and analyzed the point convergence or gathering issue in a discrete-time synchronized setting, assuming agents can see and measure both distances and bearings to neighbors within the visibility range.

Subsequently, in a series of papers, Gordon et al. (2004, 2005, 2008) analyzed gathering with limited visibility and bearing-only sensing constraints imposed on the agents. Their work proved finite time gathering or clustering results in discrete-time settings, and also proposed dynamics for the continuous time settings. In Sect. 3 of this paper, we shall discuss the continuous time motion model they analyzed and compare it to our dynamic rule of motion.

In our work, as well as in most of the papers cited above, it is assumed that the agents can directly control their velocity with no acceleration constraints. We note that the literature of multi-agent systems is replete with papers assuming more complex and realistic dynamics for the agents, like unicycle motions, second-order systems and double integration models relating the location to the controls and seek sensor-based local control-laws that ensure gathering or the achievement of some desired configuration. However, we feel that it is still worthwhile exploring systems with agents directly controlling their velocity based on very primitive sensing, in order to test the limits on what can be achieved by simple agents with reactive behaviors.

2 The gathering problem

We consider N agents located in the plane ($\mathbb{R}^2$) whose positions are given by $\{P_i = (x_i, y_i)^T\}_{i=1,2,\ldots,N}$, in some absolute coordinate frame which is unknown to the agents. We define the vectors

$$u_{ij} = \begin{cases} \frac{P_j - P_i}{\|P_j - P_i\|} & 0 < \|P_j - P_i\| \le V \\ 0 & \|P_j - P_i\| = 0 \text{ or } \|P_j - P_i\| > V \end{cases}$$

hence u_{ij} are, if not zero, the unit vectors from P_i to all P_j's which are neighbors of P_i in the sense of being at a distance less than V from P_i, i.e., P_j's obeying:

$$\|P_j - P_i\| \triangleq \left[(P_j - P_i)^T (P_j - P_i)\right]^{1/2} \le V$$

Note that we have $u_{ij} = -u_{ji}, \forall (i, j)$. For each agent P_i, let us define the special vectors, u_i^+ and u_i^- (from among the vectors u_{ij} defined above). Consider the nonzero vectors from the set $\{u_{ij}\}_{j=1,2,\ldots,N}$. Anchor a moving unit vector $\bar{\eta}(\theta)$ at P_i pointing at some arbitrary neighbor, i.e., at $u_{ik} \neq 0$, $\bar{\eta}(0) = u_{ik}$ and rotate it clockwise, sweeping a full circle around P_i. As $\bar{\eta}(\theta)$ goes from $\eta(0)$ to $\eta(2\pi)$ it will encounter all the possible nonzero u_{ij}'s and these encounters define a sequence of angles $\alpha_1, \alpha_2, \ldots, \alpha_r$ that add to $2\pi = \alpha_1 + \cdots + \alpha_r$ ($\alpha_k =$ angle from k-th to $(k+1)$-th encounter with a u_{ij}, $\alpha_r =$ angle from last encounter to first one again, see Fig. 2). If none of the angles $\{\alpha_1, \ldots, \alpha_r\}$ is bigger than π, set $u_i^+ = u_i^- = 0$. Otherwise define $u_i^+ = u_{i(m)}$ and $u_i^- = u_{i(n)}$, the unit vectors encountered when entering and exiting the angle $\alpha_b > \pi$ bigger than π.

One might call u_i^- the pointer to the leftmost visible agent from P_i and u_i^+ the pointer to the rightmost visible agent among the neighbors of P_i. If P_i has nonzero right and leftmost visible agents it means that all its visible neighbors belong to a disk sector defined by an angle less than π, and P_i will be movable. Otherwise, we call it "surrounded" by neighbors and, in this case, it will stay in place, while it remains surrounded (see "Appendix 2" for an alternative way of defining the leftmost and rightmost agents).

The dynamics of the multi-agent system will be defined as follows.

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = v_0 \left(u_i^+ + u_i^-\right) \quad \text{for} \quad i = 1, \ldots, N \tag{1}$$

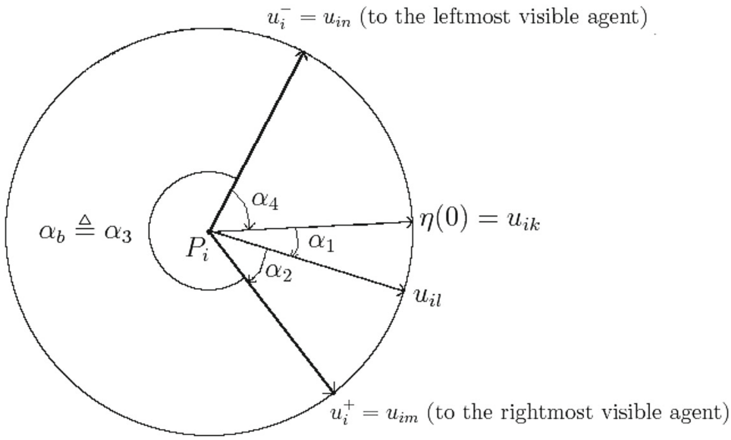


Fig. 2 *Leftmost* and *rightmost* visible agents of agent located at P_i

Note that the speed of each agent is in the span of $[0, 2v_0]$, v_0 being a positive constant, equal to half the maximal speed of the agents. With this we have defined a local, distributed, reactive law of motion based on the information gathered by each agent. Notice that the agents do not communicate directly, are all identical, and have limited sensing capabilities. Yet we shall show that, under the defined reactive law of motion, the agents will eventually gather to a point, while the distances between all pairs of visible agents monotonically decrease. Assume that we are given an initial configuration of N agents placed in the plane in such a way that their visibility graph is connected. This means that there is a path (or a chain) of mutually visible neighbors from each agent to any other agent. Our first result is that while agents move according to the above-described rule of motion, the visibility graph will only be supplemented with new edges, and old visibility connections will never be lost.

2.1 Connectivity is never lost

We next prove

Proposition 1 *A multi-agent system under the dynamics*

$$\left\{\dot{P}_i = v_0\left(u_i^+ + u_i^-\right)\right\}_{i=1,\ldots,N}$$

ensures that pairs of neighboring agents at $t = 0$ (i.e., agents at a distance less than V) will remain neighbors forever.

Proof To prove this result we shall consider the dynamics of distances between pairs of agents. We have that the distance Δ_{ij} between P_i and P_j is

$$\Delta_{ij} = \|P_j - P_i\| = \left[(P_j - P_i)^T(P_j - P_i)\right]^{1/2}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{ij}^{(t)} = \frac{1}{\|P_j - P_i\|}(P_j - P_i)^T(\dot{P}_j - \dot{P}_i)$$

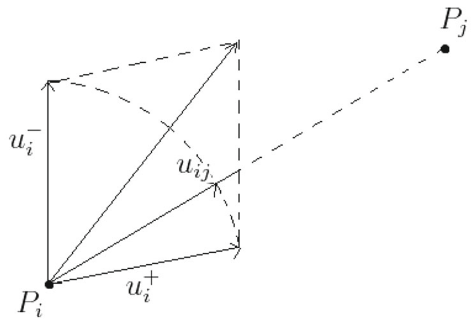


Fig. 3 $u_{ij}^T\left(u_i^+ + u_i^-\right) \geq 0$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{ij}^{(t)} = u_{ij}^T(\dot{P}_j - \dot{P}_i)$$
$$= -u_{ij}^T\dot{P}_i + u_{ij}^T\dot{P}_j$$
$$= -u_{ij}^T\dot{P}_i - u_{ji}^T\dot{P}_j$$

but we know that the dynamics (1) is

$$\dot{P}_i = v_0\left(u_i^+ + u_i^-\right)$$
$$\dot{P}_j = v_0\left(u_j^+ + u_j^-\right)$$

therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{ij}^{(t)} = -v_0u_{ij}^T\left(u_i^+ + u_i^-\right) - v_0u_{ji}^T\left(u_j^+ + u_j^-\right)$$

However, for every agent P_i we have either $u_i^+ + u_j^- \triangleq 0$ if the agent is surrounded, or $u_i^+ + u_i^-$ is in the direction of the center of the visibility disk sector in which all neighbors (including P_j) reside (see Fig. 3).

Therefore, the inner product $u_{ij}^T\left(u_i^+ + u_i^-\right)$ will necessarily be positive (see "Appendix 3" for a formal proof), hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{ij}^{(t)} = -\left(v_0u_{ij}^T\left(u_i^+ + u_i^-\right) + v_0u_{ji}^T\left(u_j^+ + u_j^-\right)\right) \leq 0$$

This shows that distances between neighbors can only decrease (or remain the same). Hence agents never lose neighbors under the assumed dynamics. □

2.2 Finite time gathering

We have shown that the dynamics of the system (1) ensures that agents that are neighbors at $t = 0$ will forever remain neighbors. We next prove that, as time passes, agents acquire new neighbors and in fact will all converge to a common point of encounter. We prove the following:

Proposition 2 *A multi-agent system with dynamics given by* (1) *gathers all agents to a point in* $\mathbb{R}^2$*, in finite time.*

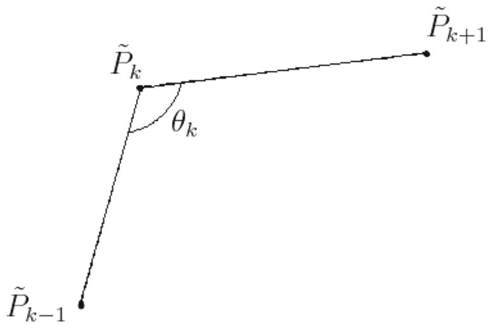


Fig. 4 Angle at a vertex of the convex hull

Proof We shall rely on a Lyapunov function $L(P_1, \ldots, P_N)$, a positive function defined on the geometry of agent constellations which becomes zero if and only if all agents' locations are identical. We shall show that, due to the dynamics of the system, the function $L(P_1, \ldots, P_N)$ decreases to zero at a rate bounded away from zero, ensuring finite time convergence. The function L will be defined as the perimeter of the convex hull of all agents' locations, $\mathrm{CH}\{P_i(t)\}_{i=1,\ldots,N}$. Indeed, consider the set of agents that are, at a given time t, the vertices of the convex hull of the set $\{P_i(t)\}_{i=1,\ldots,N}$. Let us call these agents $\{\tilde{P}_k(t)\}$ for $k = 1, \ldots, K \leq N$. For every agent $\tilde{P}_k$ on the convex hull (i.e., for every agent that is a corner of the convex polygon defining the convex hull), we have that all other agents are in a region (wedge) determined by the half lines from $\tilde{P}_k$ in the directions $\tilde{P}_k\tilde{P}_{k-1}$ and $\tilde{P}_k\tilde{P}_{k+1}$, a wedge with an opening angle θ_k (see Fig. 4).

Since clearly $\theta_k \leq \pi$ for all k, we must have that agent $\tilde{P}_k$ has all its visible neighbors in a wedge of its visibility disk with an angle $\alpha_k \leq \theta_k \leq \pi$; hence, its u_k^+ and u_k^- vectors will not be zero, causing the motion of $\tilde{P}_k$ toward the interior of the convex hull. This will ensure the shrinking of the convex hull, while it exists, and the rate of this shrinking will be determined by the evolution of the constellation of agents' locations. Let us formally prove that, indeed, the convex hull will shrink to a point in finite time. Consider $L(t)$, the perimeter of the convex hull of all agent locations $\mathrm{CH}\{P_i(t)\}_{i=1,\ldots,N}$

$$L(t) = \sum_{k=1}^{K(t)} \Delta_{k,k+1} = \sum_{k=1}^{K(t)} \left[(\tilde{P}_{k+1}(t) - \tilde{P}_k(t))^T (\tilde{P}_{k+1}(t) - \tilde{P}_k(t)) \right]^{1/2}$$

where the indices are considered modulo $K(t)$ and recall that we renamed the agents on the convex hull as $\{\tilde{P}_1(t), \ldots, \tilde{P}_K(t)\}$ (with $K \leq N$).

We have, assuming that K remains the same for a while, that

$$\frac{\mathrm{d}}{\mathrm{d}t} L(t) = \sum_{k=1}^{K} \frac{\mathrm{d}}{\mathrm{d}t} \Delta_k = -\sum_{k=1}^{K} \left(v_0 \tilde{u}_{k,k+1}^T \left(u_k^+ + u_k^- \right) + v_0 \tilde{u}_{k,k+1}^T \left(u_{k+1}^+ + u_{k+1}^- \right) \right)$$

but note that $\tilde{u}_{k,k+1}$ does not necessarily lie between u_k^+ and u_k^- anymore, since, in fact, $\tilde{P}_k$ and $\tilde{P}_{k+1}$ might not even be neighbors.

Now let us consider $\frac{\mathrm{d}}{\mathrm{d}t} L(t)$ and rewrite it as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} L(t) = -v_0 \sum_{k=1}^{K} \tilde{u}_{k,k+1}^T \left(u_k^+ + u_k^- \right) - v_0 \sum_{k=1}^{K} \tilde{u}_{k+1,k}^T \left(u_{k+1}^+ + u_{k+1}^- \right)$$

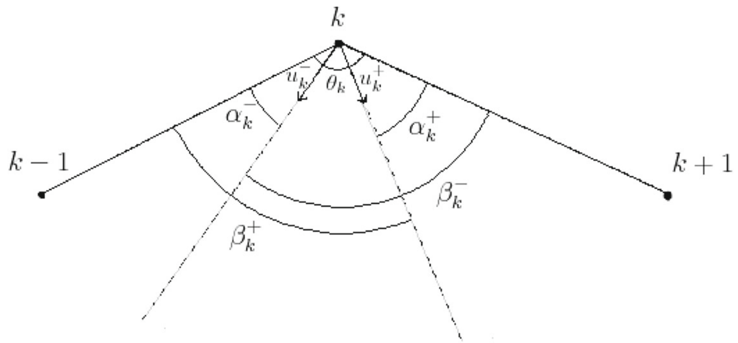


Fig. 5 Angles at a vertex of the convex hull

Rewriting the second term above, by shifting the indices k by -1 we get

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) = -v_0 \sum_{k=1}^{K} \tilde{u}_{k,k+1}^T \left(u_k^+ + u_k^-\right) - v_0 \sum_{k=1}^{K} \tilde{u}_{k,k-1}^T \left(u_k^+ + u_k^-\right)$$

This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) = -v_0 \sum_{k=1}^{K} < u_k^+, \tilde{u}_{k,k+1} + \tilde{u}_{k,k-1} > -v_0 \sum_{k=1}^{K} < u_k^-, \tilde{u}_{k,k+1} + \tilde{u}_{k,k-1} >$$

Note that we have here inner products between unit vectors, yielding the cosines of the angles between them. Therefore, defining θ_k to be the angle between $\tilde{u}_{k,k-1}$ and $\tilde{u}_{k,k+1}$ (i.e., the interior angle of the convex hull at the vertex k, see Fig. 5), and the angles:

$$\begin{aligned}
\alpha_k^+ &\triangleq \gamma\left(u_k^+, \tilde{u}_{k,k+1}\right)\\
\beta_k^+ &\triangleq \gamma\left(\tilde{u}_{k,k-1}, u_k^+\right)\\
\alpha_k^- &\triangleq \gamma\left(\tilde{u}_{k,k-1}, u_k^-\right)\\
\beta_k^- &\triangleq \gamma\left(u_k^-, \tilde{u}_{k,k+1}\right)
\end{aligned}$$

where $\gamma(u, v)$ is the angle formed by the vectors u and v.

We have $\alpha_k^+ + \beta_k^+ = \alpha_k^- + \beta_k^- = \theta_k$, and all these angles are between 0 and π.

Using these angles we can rewrite

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) = -\sum_{k=1}^{K} v_0 \left(\cos\alpha_k^+ + \cos\beta_k^+\right) - \sum_{k=1}^{K} v_0 \left(\cos\alpha_k^- + \cos\beta_k^-\right)$$

Now, using the inequality (proved in "Appendix 1")

$$\begin{aligned}
&\cos\alpha + \cos\beta \geq 1 + \cos(\alpha+\beta)\\
&0 \leq \alpha, \beta, \alpha+\beta \leq \pi
\end{aligned} \tag{2}$$

we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}L(t) \geq 2v_0 \sum_{k=1}^{K} (1 + \cos\theta_k) \tag{3}$$

□

For any convex polygon we have the following result (see the detailed proof in "Appendix 1"):

Geometric Lemma 1 *For any convex polygon with K vertices and interior angles $\theta_1, \ldots, \theta_K$, with $\theta_1 + \cdots + \theta_K = (K-2)\pi$ we have that*

$$\sum_{k=1}^{K} \cos(\theta_i) \geq \begin{cases} 1 + (K-1)\cos\left(\frac{(K-2)\pi}{K-1}\right), & 2 \leq K \leq 6 \\ K\cos\left(\frac{(K-2)\pi}{K}\right), & K \geq 7 \end{cases} \tag{4}$$

Therefore, we obtain from (3) and (4) that

$$-\frac{\mathrm{d}}{\mathrm{d}t}L(t) \geq \mu(K) \tag{5}$$

where

$$\begin{aligned} \mu(K) &= 2v_0\left(K + \left\{\begin{matrix} 1 + (K-1)\cos\left(\frac{(K-2)\pi}{K-1}\right) & 2 \leq K \leq 6 \\ K\cos\left(\frac{(K-2)\pi}{K}\right) & K \geq 7 \end{matrix}\right\}\right) \\ &= 2v_0K\left(1 - \max\left\{\cos\left(\frac{2\pi}{K}\right), \frac{K-1}{K}\cos\left(\frac{\pi}{K-1}\right) - \frac{1}{K}\right\}\right) \end{aligned}$$

Note here that, since $(1 - \max\{\ldots\}) > 0$ we have that the rate of decrease in the perimeter of the configuration is strictly positive, while the convex hull of the agents' locations is not a single point.

The argument outlined so far assumed that the number of agents determining the convex hull of their constellation is a constant K. However, in the course of evolution some agents may collide and merge and/or some agents become exposed as vertices of the convex hull. Hence, the integer value of K may vary in time. At a collision between two agents we assume that they merge and thereafter continue to move as a single agent. However, we have that whatever is the value of K, the perimeter decreases at a positive rate, and this rate is bounded away from zero. Therefore, we have effectively proved that the perimeter of the convex hull will necessarily vanish in finite time.

Figure 6 shows the bound on the rate of decrease of $L(t)$, as a function of K assuming $v_0 = 1$. Note that we always have $K \leq N$, and $\mu(K)$ is a decreasing function of K; hence, for any system with a finite number of agents N there will be a strictly positive constant $\mu(N)$ so that

$$-\frac{\mathrm{d}}{\mathrm{d}t}L(t) \geq \mu(N)$$

This ensures that after a finite time, T_{ub}, given by

$$\mu(N)T_{\mathrm{ub}} = L(0) \Rightarrow T_{\mathrm{ub}} = \frac{L(0)}{\mu(N)}$$

we shall have $L(T_{\mathrm{ub}}) = 0$.

Hence, we have an upper bound on the time of convergence for any configuration of N agents given by $\frac{L(0)}{\mu(N)}$. This concludes the proof of Proposition 2.

Note from (3) that the effective rate of decrease does not depend on the perimeter of the convex hull but only on the number of agents forming it. This was an expected result, since the dynamics does not rely on Euclidean distances. This bound is decreasing with K, so that the more agents form the convex hull, the smaller will be the rate of decrease. For $K = 2$ and $K = 3$ the bound is $8v_0$, for $K = 4$, it is $7v_0$, and then it keeps decreasing for higher values

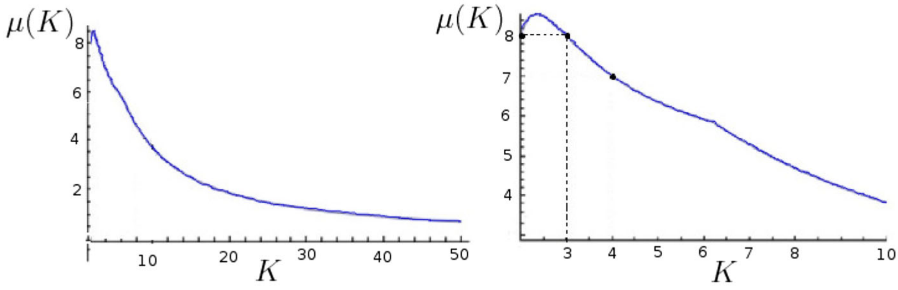


Fig. 6 Graph of the bound $\mu(K)$ of (3). The *graph* on the *right* is a zoom on small values of K

of K, slowly converging toward 0. Note the change of curve between $K = 6$ and $K = 7$, due to the discontinuity in the geometric result exhibited in Eq. (1).

The inequalities of (2) and of (4) become equalities for particular configurations of the agents. For example, in the case of a regular polygon in which each pair of adjacent neighbors is visible to each other, if $K \geq 7$, the bound in (3) yields the exact rate of convergence to zero of the convex hull perimeter.

3 Generalizations

The above analysis can be generalized to dynamics of the form

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = f(P^{(i)})\left(u_i^+ + u_i^-\right) \quad \text{for} \quad i = 1, \ldots, N \tag{6}$$

where $f\left(P^{(i)}\right) \geq 0$ is some positive function of the configuration of the neighbors, as seen by agent i. This generalization also guarantees that the rule of motion is locally defined and reactive, and defined in the same way for all agents. The dynamics (1) corresponds to a particular case of (6), with $f\left(P^{(i)}\right) = v_0$, a constant for all agents.

It is easy to slightly change the proofs above in order to show that Proposition 1 (ensuring that connectivity is not lost) is still valid as long as $f(P^{(i)}) \geq 0$ for all i, and that Proposition 2 (ensuring finite time gathering) is also valid as long as $f(P^{(i)}) \geq \epsilon > 0$ for all i.

Note that in the work of Gordon et al. (2004), a constant speed for the agents was considered, and this corresponds to setting $f(P^{(i)}) = \frac{1}{\|u_i^+ + u_i^-\|}$, rather than v_0, for a mobile agent i. Given that, in this case, $f\left(P^{(i)}\right) \geq \frac{1}{2}$, the conditions for Propositions 1 and 2 are verified, and hence the dynamics with constant speed also ensures convergence to a single point, with pairs of initially visible agents never losing connectivity. Therefore, we also have a proof for the convergence of the algorithm that was proposed in the above-mentioned paper.

However, it was pointed out in the above-mentioned paper that in the model with constant speed, one has to deal with quite unpleasant, chattering effects that were called "Zenoness-effects" (Gordon et al. 2004) in order to effectively modulate the speed of motion of some of the agents. Zenoness is the effect of having an infinite number of switches between unit and zero speed—in a finite interval. In certain configurations, an agent may oscillate infinitely often between a position in which it is surrounded, to a position in which it is not. This implies alternating its speed infinitely often between zero and a constant value. In contrast to the constant speed model of Gordon et al. (2004), in our model, the speed varies smoothly in the range $[0, 2v_0]$, with the clear advantage of having a naturally controlled modulation on the speed of the agents.

Recently, Kempkes et al. (2012) and Degener et al. (2015) have analyzed the continuous time system proposed by Gordon et al. (2004) with constant speed of agents and proved by a different method finite time convergence, providing another bound on the time to gathering. The bound they found is a fixed constant bound on the rate of decrease of a different geometric measure they define and call the "outer perimeter" of the configuration. We note that this quantity is not the perimeter of the convex hull, the Lyapunov function we propose herein. In the case of constant speed agents, a comparison of their bound to ours calls for a careful consideration of the geometric constellation of the agents and would be an interesting topic for further research.

4 Simulations

Let us consider some simulations of the multi-agent dynamics discussed in this paper. We start with a randomly generated swarm and then we consider some interesting special con-

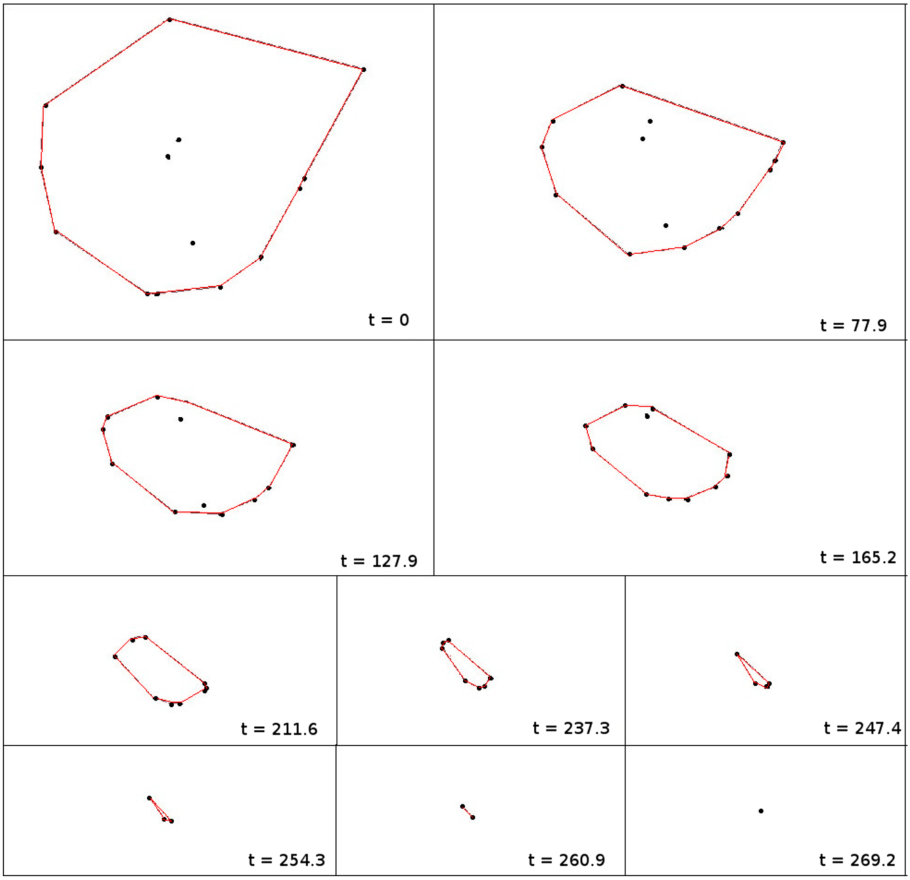


Fig. 7 Configuration of the swarm at different times, for a randomly selected initial configuration. The convex hull of the set of agents is also displayed

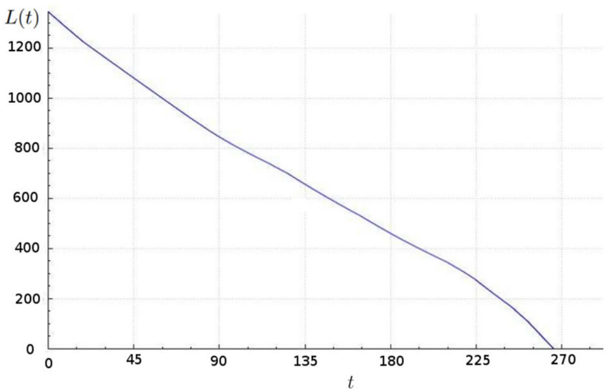


Fig. 8 The perimeter of the convex hull $L(t)$, decreasing until it reaches zero

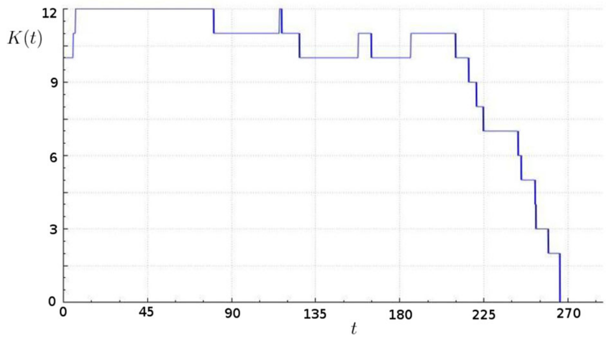


Fig. 9 Number of agents $K(t)$ defining the convex hull of the set. The number $K(t)$ decreases when two or more agents of the hull merge and increases when one or more agents join the perimeter of the hull

figurations. The simulations were written in C++, the user interface being designed using the Qt library. Simulation of continuous time was achieved by selecting timesteps sufficiently small with respect to the magnitudes of the systems parameters.

4.1 Randomly selected configuration of 15 agents

First, we randomly generate a swarm of 15 agents, with a connected visibility graph as initial configuration. We arbitrarily set v_0 to 1 and visibility to 200. The configuration of the swarm at different times during the evolution is given in Fig. 7.

We also plot some properties of the swarm during its evolution in time. Figure 8 represents the perimeter of the convex hull of the set of agents. Figure 9 plots the count of indistinguishable agents (i.e., collided agents count as one) in the convex hull's perimeter. Figure 10 represents the time derivative of the convex hull perimeter and the theoretical bound given by Eq. (5), which is a function of the number of indistinguishable agents forming the convex hull's perimeter.

For a fixed number of agents forming the convex hull, one can see in Fig. 10 that the derivative of the perimeter tends to the theoretical bound. This can be intuitively explained by the fact that far away agents evolve toward the inside more rapidly, making the con-

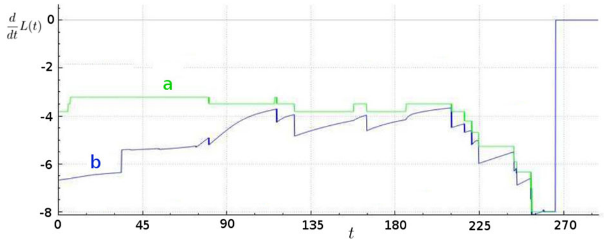


Fig. 10 Rate of change (the negative derivative) of the convex hull perimeter (*blue*, labeled *b*) compared to theoretical bound (*green*, labeled *a*), which is a function of the number of agents defining the current convex hull, *K*, as given by formula (5)

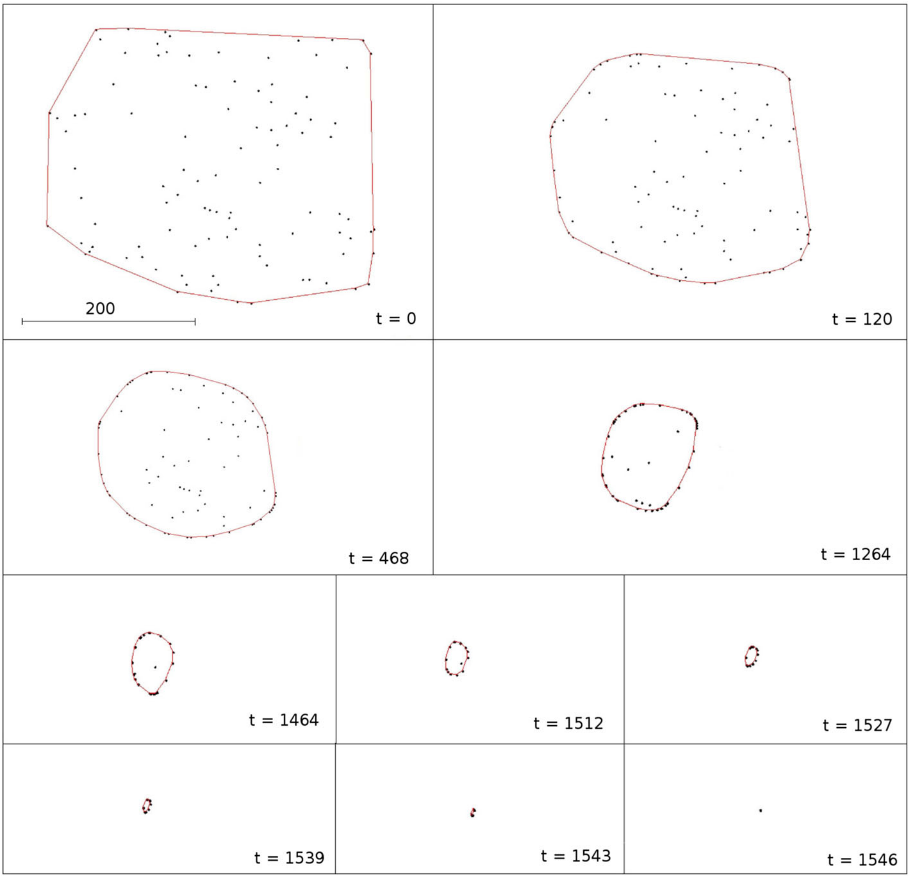


Fig. 11 Configuration of the swarm at different times, for a randomly selected initial configuration of 100 agents. The temporal evolution of the convex hull of the set of agents is also displayed

vex hull shape more rounded, approaching the regular shapes that yield the theoretical bound.

The discontinuity of the derivative of the perimeter of the convex hull that occurs when there is no change in the number of agents forming the hull, for example around $t = 35$ in Fig. 10, is due to change in the connectivity graph (which has not been printed for clarity).

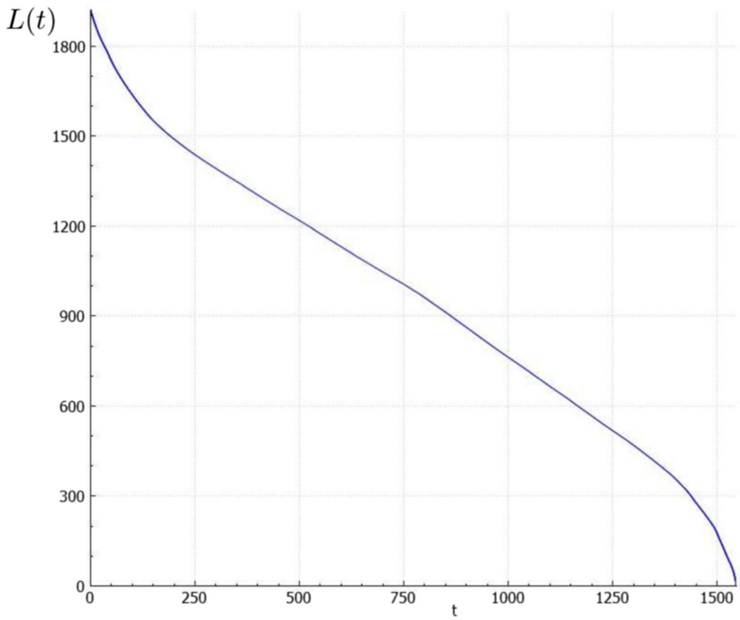


Fig. 12 Perimeter of the convex hull for a swarm of 100 agents, decreasing until it reaches zero

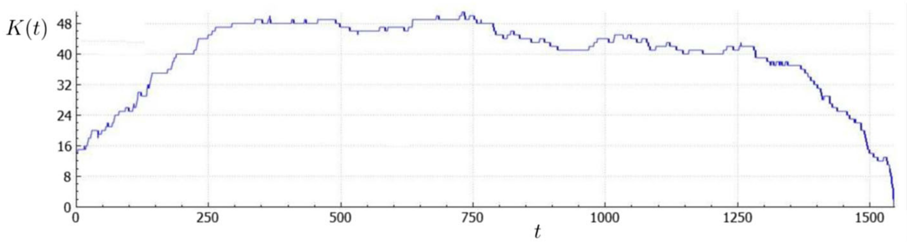


Fig. 13 Number of agents defining the convex hull for a swarm of 100 agents

In this particular case, two agents on the top left became visible to each other at this time and their directions and speed changed, slowing down slightly the rate of perimeter decrease.

4.2 Randomly selected configuration of 100 agents

We randomly generated a swarm of 100 agents, having a connected visibility graph in the initial configuration. The v_0 and the visibility range are set as in the previous case. The configuration of the swarm at different times during its evolution is given in Fig. 11.

The perimeter of the convex hull of the set of agents, the count of indistinguishable agents defining the convex hull, and the time derivative of the convex hull perimeter along with the theoretical bound given by Eq. (5) are given in Figs. 12, 13, and 14.

In this case, due to the increased number of agents, the effect of the swarm constellation approaching a regular formation is more visible.

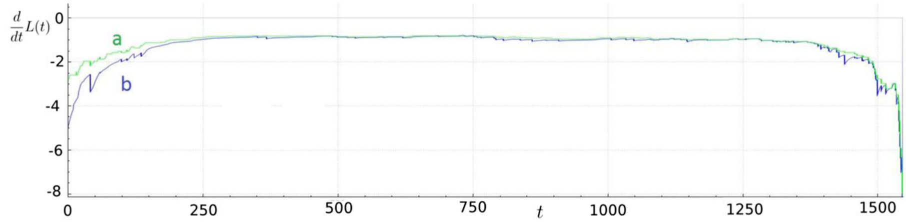


Fig. 14 Rate of change (derivative) of the convex hull perimeter (*blue*, labeled *b*) compared to theoretical bound (*green*, labeled *a*), for a swarm of 100 agents

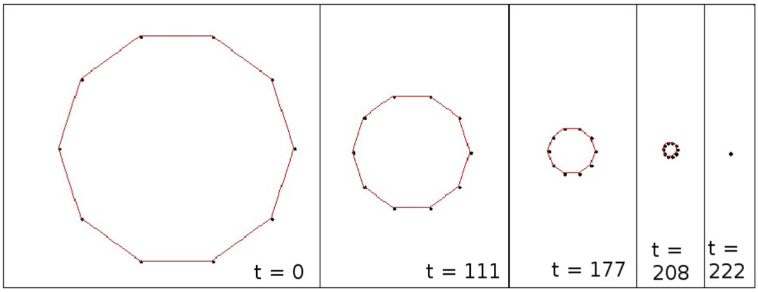


Fig. 15 Starting from a regular *polygon*, the swarm keeps its regular *shape* until simultaneously gathering to a point. The *leftmost* and *rightmost* neighbors of each agent are, in this case, its two neighbors in the *polygon*

4.3 Regular polygon constellations

An initial configuration that is a regular polygon with 10 agents was simulated here. Again, v_0 is set to 1 and visibility is 200. As expected, the rate of decrease of the perimeter of the convex hull is constant, and all the agents contribute to the convex hull all along, until the very end where they collide and merge. Simulation results and the converging parameters' plots are shown in Figs. 15 and 16.

4.4 Close-to-minimum configuration with $n = 4$

We start at a configuration close to the one that reaches the minimum of the sum of the cosines of the interior angles of the polygon. Results are presented in Figs. 17, 18 and 19. One can see that this configuration provides a smaller decreasing rate than the configuration of a regular polygon, in conformity with our analysis. The bound is not attained for a regular polygon configuration for small values of n.

5 Concluding remarks

We have shown that a very simple local control on the velocity of point agents in the plane, based on limited visibility and bearing-only sensing of neighbors, ensures their finite time gathering. The motion rule is simple. The agents' velocity is set to the vector sum of two unit vector pointers. These point to the two extremal neighbors if all visible neighbors reside inside a half plane (a half-disk) around the agent. Otherwise, the velocity is set to zero. This very simple rule of behavior is different from the one assumed by Gordon et al. (2004), where

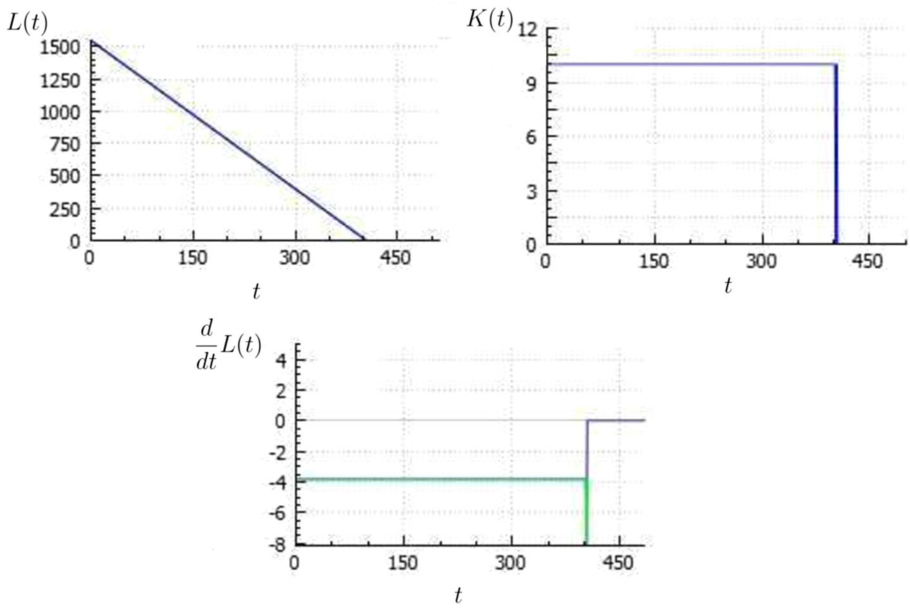


Fig. 16 *Top left* the perimeter of the convex hull decreases at a constant rate. *Top right* the number of agents defining the convex hull is constant. *Bottom* the derivative of the perimeter of the convex hull is constant and equals the theoretical bound all along the evolution

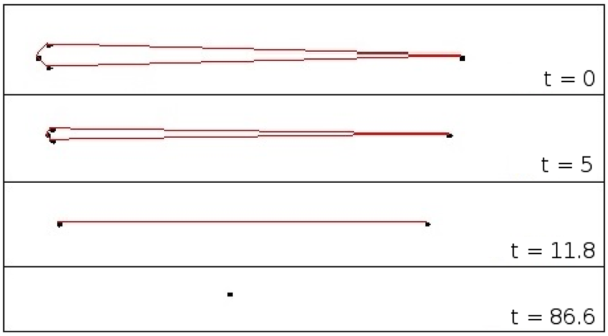


Fig. 17 A configuration in the form of a cone, close to one of the theoretical configurations that yield the $\mu(K)$ bound for $2 \leq K \leq 6$ (here close to the state in Fig. 23 of "Appendix 1," that yields the bound $\mu(4)$). In this case, the gathering can be divided in two stages. First, the three agents of one side merge, then the resulting composite agent merges with the single agent of the far side

the motion was set to have a constant velocity in the direction bisecting the disk sector where visible neighbors reside, or zero if the agent was "surrounded." As we show in this paper that model and several other possible generalizations also ensure gathering. However, the proposed constant-velocity model had to deal with chattering, or Zenoness-effects in order to effectively modulate the speed of motion of agents, as discussed in Gordon et al. (2004). Zeno effects involve infinitely many switches between motion at speed one and at speed zero in a finite interval. Two interesting recent papers by Kempkes et al. (2012) and Degener et al. (2015) carefully analyzed the process of finite time gathering of robots acting according to

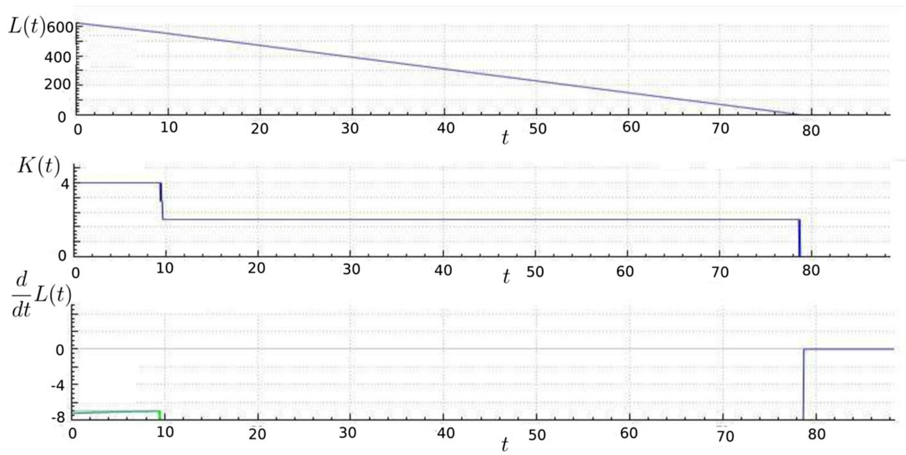


Fig. 18 From *top* to *bottom* the perimeter of the convex hull, the number of agents defining it, and its derivative

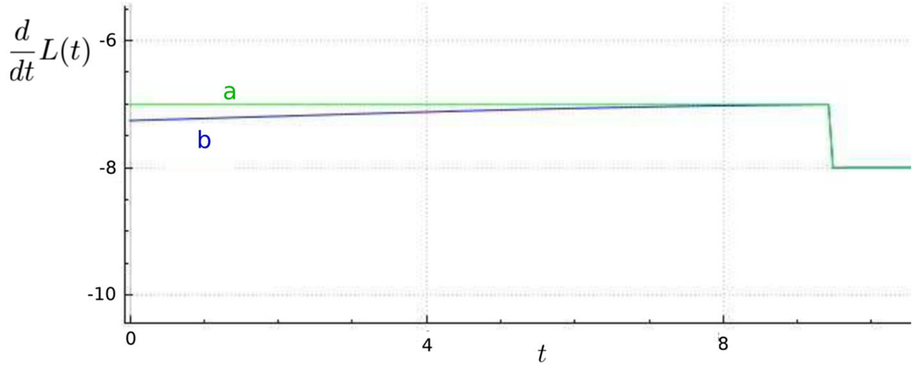


Fig. 19 A zoom on the derivative of the perimeter of the convex hull (in *blue*, labeled *b*) at the beginning of the dynamics. A regular polygon (a *square* in this case) would give a constant rate of -8, lower than the current rate approaching the theoretical bound of -7 (in *green*, labeled *a*)

the interaction rule defined in Gordon et al. (2004), and gave nice, explicit bounds on the time to gathering.

In this paper, in conjunction with our model, and some generalizations too, including the model of Gordon et al. (2004), we provided a very simple geometric proof that finite time gathering is achieved, and also provided precise bounds on the rate of decrease of the perimeter of the agent configuration's convex hull. These bounds are based on a novel geometric lower bound on the sum of cosines of the interior angles of an arbitrary convex planar polygon. This geometrical result is interesting in its own right, and one can notice that the bound is different for polygons with less than 7 vertices from those with more than 7 vertices. Our result may be regarded as a convergence proof for a highly nonlinear autonomous dynamic system, naturally handling dynamic changes in its dimension (the events when two agents meet and merge).

Appendix 1: Proof of Lemma 1

We shall first prove the following fact

Inequality a *Let* $0 \le a \le b \le \pi$ *and* $0 \le a+b \le \pi$. *Then we have*

$$\begin{aligned}\sqrt{2(1+\cos(a+b))} &= 2\cos\left(\frac{a+b}{2}\right) \ge \cos(a)+\cos(b) \ge 2\cos^2\left(\frac{a+b}{2}\right)\\ &= 1+\cos(a+b)\end{aligned}$$

Proof The function cosine is decreasing in $[0, \pi]$, and given that $\frac{a+b}{2} \ge \frac{b-a}{2}$:

$$1 \ge \cos\left(\frac{b-a}{2}\right) \ge \cos\left(\frac{a+b}{2}\right)$$

multiplying by $2\cos\left(\frac{a+b}{2}\right) \ge 0$:

$$\begin{aligned}2\cos\left(\frac{a+b}{2}\right) &\ge 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{b-a}{2}\right) \ge 2\cos^2\left(\frac{a+b}{2}\right)\\ 2\cos\left(\frac{a+b}{2}\right) &\ge \cos(a)+\cos(b) \ge 1+\cos(a+b)\end{aligned}$$

□

A direct consequence is the following result.

Inequality a *Let* $0 \le a, b \le \pi$. *Then*

$$\cos(a)+\cos(b) \ge \begin{cases} 1+\cos(a+b): & a+b \le \pi \\ 2\cos\left(\frac{a+b}{2}\right): & a+b \ge \pi \end{cases}$$

Proof The first line is already part of Lemma 1. The second line can be proven by using the left inequality of Lemma 1 with $\pi - a$ and $\pi - b$, noticing that $0 \le \pi - a \le \pi$, $0 \le \pi - b \le \pi$, and $\pi - a + \pi - b \le \pi$ for $a + b \ge \pi$. □

We proceed to prove Lemma 1. Suppose any given initial configuration of the polygon with interior angles $0 \le x_1, \ldots, x_n \le \pi$. We then have $x_1 + \cdots + x_n = (n-2)\pi$.

Consider all pairs of nonzero values (x_i, x_j). As long as there is still a pair obeying $x_i + x_j \le \pi$, transform it from (x_i, x_j) to $(0, x_i + x_j)$. When there are no such pairs, among all the nonzero values, take the minimum value and the maximum value, x_i and x_j (they must verify $x_i + x_j \ge \pi$ due to the previously applied process), and transform the pair from (x_i, x_j) to $\left(\frac{x_i+x_j}{2}, \frac{x_i+x_j}{2}\right)$.

Repeat the above process. We prove that the process converges, and that we approach a configuration where all nonzero values are equal. Note that after each step, the sum of the values equals $(n-2)\pi$, and the values of all x_i's remain between 0 and π.

The number of values that the above process sets to zero must be less or equal to 2 in order to have the sum of the n positive values equal to $(n-2)\pi$. Therefore, it is guaranteed that after a finite number of iterations, there will be no pairs of nonzero values whose sum is less than π (otherwise, this would allow us to add a zero value without changing the sum).

Once in this situation, we replace pairs of farthest nonzero values (x_i, x_j) with the pair $\left(\frac{x_i+x_j}{2}, \frac{x_i+x_j}{2}\right)$. Let us show that all the nonzero values converge to the same value, specifically to their average.

Let k be the number of remaining nonzero values after the iteration t_0 which sets the last value to zero. Denote these values at the i-th iteration by $\left(x_1^{(i)}, \ldots, x_k^{(i)}\right)$. Define:

$$m = \frac{x_1^{(i)} + \cdots + x_k^{(i)}}{k} = \frac{(n-2)\pi}{k}$$
$$E_i = \left(x_1^{(i)} - m\right)^2 + \cdots + \left(x_k^{(i)} - m\right)^2$$

Without loss of generality, suppose that at the i-th iteration the extreme values were x_1 and x_2 and we mapped $\left(x_1^{(i)}, x_2^{(i)}\right)$ to $\left(x_1^{(i+1)} = \frac{x_1^{(i)}+x_2^{(i)}}{2}, x_2^{(i+1)} = \frac{x_1^{(i)}+x_2^{(i)}}{2}\right)$. Now we have:

$$E_{i+1} - E_i = 2\left(\frac{x_1^{(i)} + x_2^{(i)}}{2} - m\right)^2 - \left(x_1^{(i)} - m\right)^2 - \left(x_2^{(i)} - m\right)^2)$$
$$= -\frac{1}{2}\left(x_1^{(i)} - x_2^{(i)}\right)^2$$

Since $x_1^{(i)}$ and $x_2^{(i)}$ are the extreme values, we have for any $1 \leq l \leq k$:

$$\left(x_1^{(i)} - x_2^{(i)}\right)^2 \geq \left(x_l^{(i)} - m\right)^2$$

and by summing over l this implies:

$$k\left(x_1^{(i)} - x_2^{(i)}\right)^2 \geq E_i$$

Hence,

$$E_{i+1} - E_i = -\frac{1}{2}\left(x_1^{(i)} - x_2^{(i)}\right)^2 \leq -\frac{E_i}{2k}$$
$$E_{i+1} \leq \left(1 - \frac{1}{2k}\right) E_i$$
$$0 \leq E_i \leq \left(1 - \frac{1}{2k}\right)^{i-t_0} E_{t_0}$$

proving that E_i converges to zero. This shows that all nonzero values converge to m.

At each step of the above-described process, according to inequality 1, the sum of cosines can only decrease. Therefore, from any given configuration we can get as close as possible to a configuration in which all nonzero values are equal, without increasing the sum of the cosines. Hence, the minimum value must be reached in a configuration in which all nonzero values are equal.

Since there can be at most only 2 zero values, the minimum value of the sum of the cosines is the minimum of the following values:

- $2 + (n-2)\cos\left(\frac{(n-2)\pi}{n-2}\right) = -(n-4)$ (case with 2 zeros)
- $1 + (n-1)\cos\left(\frac{(n-2)\pi}{n-1}\right)$ (case with 1 zero)
- $n\cos\left(\frac{(n-2)\pi}{n}\right)$ (case with no zero)

Let us compare these values. Define the function $e(x) = \cos(x) + x\sin(x)$ for $0 \leq x \leq \pi$. Basic calculations give us $e^{'}(x) = x\cos(x)$ and therefore e is increasing in $\left[0, \frac{\pi}{2}\right]$ and decreasing in $\left[\frac{\pi}{2}, \pi\right]$. Therefore for $0 \leq x \leq \frac{\pi}{2}$, $e(x) \geq e(0) = 1$

In order to compare the case with 2 zeros to the case with 1 zero, define for $n \geq 2$

$$\begin{aligned} f(n) &= 1 + (n-1)\cos\left(\frac{(n-2)\pi}{n-1}\right) + (n-4) \\ &= n - 3 - (n-1)\cos\left(\frac{\pi}{n-1}\right) \\ f^{'}(n) &= 1 - \cos\left(\frac{\pi}{n-1}\right) - \frac{\pi}{n-1}\sin\left(\frac{\pi}{n-1}\right) \\ &= 1 - e\left(\frac{\pi}{n-1}\right) \end{aligned}$$

Therefore $f^{'}(n) \leq 0$ for $n \geq 4$, because $\frac{\pi}{n-1} \leq \frac{\pi}{2}$

$$\begin{aligned} &f(2) = 0 \leq 0 \\ &f(3) = 0 \leq 0 \\ &f(4) = -\frac{1}{2} \leq 0 \\ &f(n) \leq f(4) \leq 0 \quad \text{for} \quad n \geq 4 \end{aligned}$$

Hence, $f(n) \leq 0$ and the case with 2 zeros is never the optimal solution (since the case with 1 zero always has a smaller or equal value).

In order to compare the two remaining values, define, for $n \geq 2$,

$$h(n) = n\cos\left(\frac{2\pi}{n}\right) - (n-1)\cos\left(\frac{\pi}{n-1}\right) + 1$$

The derivative w.r.t. to n is

$$\begin{aligned} h^{'}(n) &= \cos\left(\frac{2\pi}{n}\right) + \frac{2\pi}{n}\sin\left(\frac{2\pi}{n}\right) - \cos\left(\frac{\pi}{n-1}\right) - \frac{\pi}{n-1}\sin\left(\frac{\pi}{n-1}\right) \\ &= e\left(\frac{2\pi}{n}\right) - e\left(\frac{\pi}{n-1}\right) \end{aligned}$$

For $n \geq 4$, we have $\frac{\pi}{n-1} \leq \frac{2\pi}{n} \leq \frac{\pi}{2}$ and therefore $h^{'}(n) \geq 0$. One can check that $h(n) \leq 0$ for $2 \leq n \leq 6$, and $h(7) > 0$, therefore $h(n) > 0$ for $n \geq 7$. This allows us to conclude that for $2 \leq n \leq 6$ the minimal configuration is the one corresponding to 1 zero, whereas for $n \geq 7$, it is the one with no zeros. This proves Lemma 1.

Let us consider the geometric interpretation of Lemma 1 by some illustrations. Consider a convex polygon of n vertices. The sum of interior angles in a convex polygon with n vertices equals to $(n-2)\pi$ and each angle is between 0 and π. Denote by C_n the bound given by Lemma 1.

$$\sum_{k=1}^{K} \cos(\theta_i) \geq \begin{cases} 1 + (K-1)\cos\left(\frac{(K-2)\pi}{K-1}\right), & 2 \leq K \leq 6 \\ K\cos\left(\frac{(K-2)\pi}{K}\right), & K \geq 7 \end{cases} \tag{7}$$

First let us notice that sometimes the minimum value of C_n corresponds to a set of interior angles that cannot be realized by a polygon in the plane. We can show that in a convex polygon, if one of the interior angles is 0, then there are exactly two interior angles of 0 and $(n-2)$ interior angles of π. The configuration that realizes the minimum of C_4 with $n = 4$ does not correspond to a realizable configuration, see Fig. 23. However, we can get arbitrarily close to this value of C_n by increasing zero angles to ϵ and subtracting from nonzero angles the values we added (see Figs. 21 and 24).

Fig. 20 Theoretical configuration corresponding to the minimum value of $C_3 = 1$

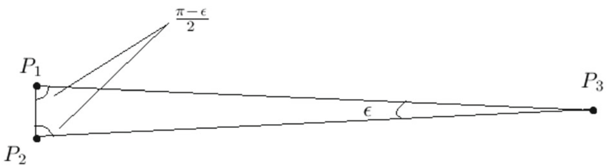
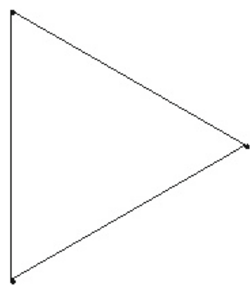


Fig. 21 Practical configuration for $n = 3$ providing a value that gets arbitrarily close to the theoretical minimum, as ϵ tends to zero

Fig. 22 Regular polygon with $n = 3$, the sum of cosines of the interior angles is $3\cos\left(\frac{\pi}{3}\right) = \frac{3}{2}$. This value is greater than the minimum possible value



If we want to minimize the cosine of an angle, we should “open” the angle maximally toward π. But the constraint of forming a convex polygon forces one to close the loop of the polygon. This is the constraint on how much we can open the angles. The more agents there are in the system, the more freedom we have to open the angles, using the many agents at our disposal to close the loop. The limiting case when there are an infinite number of agents corresponds to a circle, and all angles can be opened at the maximum of π.

What Lemma 1 implies is that, surprisingly, the configuration of the polygon that reaches the minimum value is a regular polygon only for $n \geq 7$, and that for $n \leq 6$, the minimum value of the sum of the cosines of the interior angles is arbitrarily closely approached by a polygon having the shape of a cone. Figures 20, 21 and 22 illustrate this for the case $n = 3$, while Figs. 23, 24, and 25 illustrate this for the case $n = 4$. The cases $n = 5$ and $n = 6$ are similar. For $n = 7$ and above, the configuration of the minimum is a regular polygon leading to the value of $C_n = n\cos\left(\frac{(n-2)\pi}{n}\right)$ (see examples in Fig. 26).

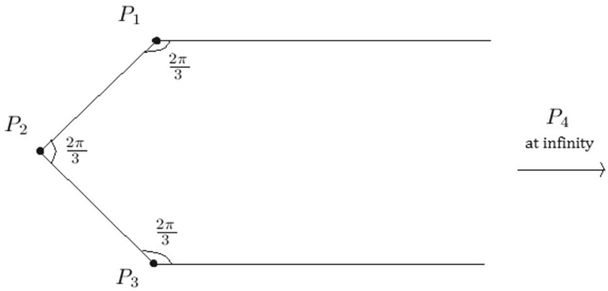


Fig. 23 Theoretical configuration corresponding to the minimum value of $C_4 = -\frac{1}{2}$

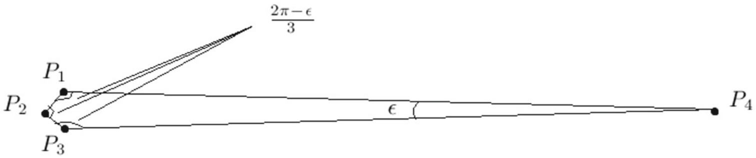


Fig. 24 Practical configuration for $n = 4$ providing a value that can get potentially arbitrarily close to the theoretical minimum of $C_4 = -\frac{1}{2}$, with $\epsilon \to 0$

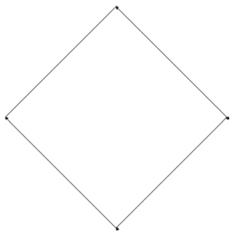


Fig. 25 Regular polygon with $n = 4$, the sum of cosines of the interior angles is $4\cos\left(\frac{2\pi}{4}\right) = 0$. This value is significantly higher than the minimum possible value

Appendix 2: Formal definitions of $u_i\pm$

The dynamics of the system described by (1) can be defined in the following alternative way.

Define the two following functions:

$$h^+(x) = \begin{cases} 1: & x > 0 \\ \frac{1}{2}: & x = 0 \\ 0: & x < 0 \end{cases}$$

and $h^-(x) = h^+(-x)$.

Let e_z be a unitary vector orthogonal to the plane (in any direction). Then define:

$$s_{ijk} = (u_{ij} \times u_{ik}) \cdot e_z$$

$$p_{ij}^{\pm} = \prod_k h^{\pm}(s_{ijk})$$

$$w_{ij}^{\pm} = \frac{p_{ij}^{\pm}}{\sum_j p_{ij}^{\pm}}$$

where $w_{ij}^{\pm} = 0$ if $\sum_j p_{ij}^{\pm} = 0$.

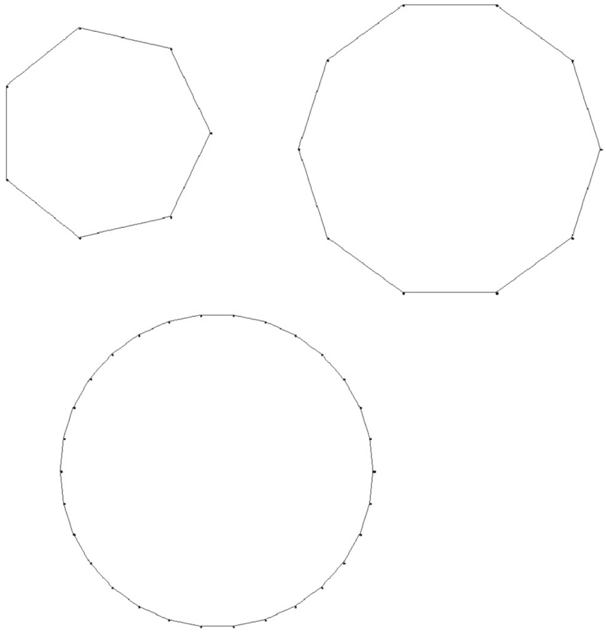


Fig. 26 *Top left* $n = 7$ is the first value of n for which the configuration of minimum value of the sum of the cosines of the interior angles corresponds to a regular *polygon*. In this case, this value is $7\cos\left(\frac{5\pi}{7}\right) \approx -4.36$. *Top right* regular *polygon* and configuration of minimum value for $n = 10$, with value $10\cos\left(\frac{8\pi}{10}\right) \approx -8.09$, each angle contributes a value of $\cos\left(\frac{8\pi}{10}\right) \approx -0.81$. *Bottom* regular polygon and configuration of minimum value for $n = 30$, with value $30\cos\left(\frac{28\pi}{30}\right) \approx -29.34$, each angle contributes a value of $\cos\left(\frac{28\pi}{30}\right) \approx -0.98$. This value gets closer to -1 with higher values of n as explained

Finally, we define $w_{ij} = w_{ij}^+ + w_{ij}^-$ and the equations of motion are given by

$$\dot{x}_i = v_0 \sum_j w_{ij} u_{ij}$$

The vectors u_i^+ and u_i^- of (1) are formally given by:

$$u_i^\pm = \sum_j w_{ij}^\pm u_{ij}$$

Appendix 3: Proof that $\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{ij}^{(t)} \leq 0$

Using the fact that u_i^- and u_{ij} are, by their definition, on the same half plane of u_i^+ yields:

$$\left(u_i^+ \times u_{ij}\right) \cdot \left(u_i^+ \times u_i^-\right) \geq 0$$

where $\times$ is the cross-product of vectors. In the same way, u_{ij} and u_i^+ are on the same half plane of u_i^-

$$\left(u_i^- \times u_{ij}\right) \cdot \left(u_i^- \times u_i^+\right) \geq 0$$

But, using the fact that these vectors are unit vectors, we get:

$$\left(u_i^+ \times u_{ij}\right) \cdot \left(u_i^+ \times u_i^-\right) = u_{ij} \cdot u_i^- - \left(u_i^+ \cdot u_i^-\right)\left(u_{ij} \cdot u_i^+\right)$$
$$\left(u_i^- \times u_{ij}\right) \cdot \left(u_i^- \times u_i^+\right) = u_{ij} \cdot u_i^+ - \left(u_i^- \cdot u_i^+\right)\left(u_{ij} \cdot u_i^-\right)$$

Therefore,

$$\left(u_i^+ \times u_{ij}\right) \cdot \left(u_i^+ \times u_i^-\right) + \left(u_i^- \times u_{ij}\right) \cdot \left(u_i^- \times u_i^+\right) = \left(1 - u_i^+ \cdot u_i^-\right) u_{ij} \cdot \left(u_i^+ + u_i^-\right) \quad (8)$$

Here, $u_i^+ \cdot u_i^- = 1$ implies that $u_i^+ = u_i^- = u_{ij}$, and in this case

$$\left(u_i^+ + u_i^-\right) \cdot u_{ij} = 2 > 0.$$

In any other case, $1 - u_i^+ \cdot u_i^- > 0$. Given that the left-hand side of (8) is positive, we must have

$$\left(u_i^+ + u_i^-\right) \cdot u_{ij} \geq 0$$

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