Blind Approximation of Planar Convex Sets

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Abstract—**The process of learning the shape of an unknown convex planar object through an adaptive process of simple measurements called Line probings, which reveal tangent lines to the object, is considered. A systematic probing strategy is suggested and an upper bound on the number of probings it requires for achieving an approximation with a pre-specified precision to the unknown object is derived. A lower bound on the number of probings required by any strategy for achieving such an approximation is also derived, showing that the gap between the number of probings required by our strategy and the number of probings required by the optimal strategy is a logarithmic factor in the worst case. The proposed approach overcomes deficiencies of the classical geometric probing approach which is based on the polygonality assumption, and thus is not applicable for real robotic tasks.**

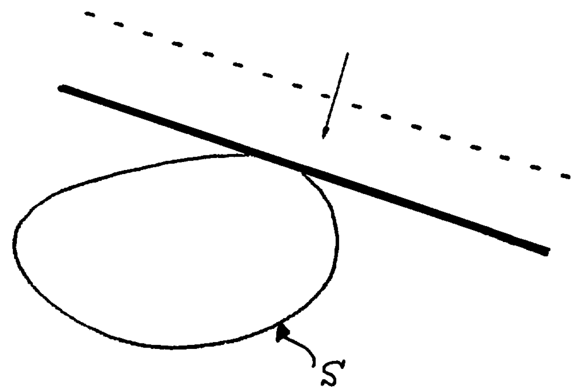

Fig. 1. Line probing.

I. Introduction

In this paper we consider the process of learning the shape of an unknown convex planar object through an adaptive process of probing. A probing is done by choosing a direction on the plane, and moving a line perpendicular to this direction, from infinity until it touches the object. Each such line probing reveals a tangent line to the object (see Fig. 1). It is clear that, in general, it will not be possible to reconstruct the object precisely from a finite sequence of such measurements. Thus, it is required to find a systematic procedure which guarantees that, after a given number of probings, the best possible approximation to the unknown object may be generated.

The problem is suggested as a simplified theoretic model for the robotic task of learning about an object from tactile sensors. It is related to the class of geometric probing problems addressed in the literature, but, as we shall see, it is very different from them. The interest in geometric probing problems was initiated by the work of Cole and Yap [4] who suggested to formulate the learning process in an algorithmic setting. They assumed the unknown planar object to be a convex polygon with unknown number of vertices, V. The sensing process was modeled as a sequence of simple measurements called *Finger probings* each done by choosing a straight directed line, and moving a point on this line, from infinity, until it touches the object. The position of the detected boundary point is the data provided by this probing. The aim defined by Cole and Yap was to find an adaptive strategy for choosing the sequence of probings, that guarantees precise reconstruction of the polygon after a minimal number of probings. Cole and Yap suggested a strategy which, under certain assumptions, guarantees the reconstruction after no more than $3V$ probings. They also derived a lower bound of $3V$ on the number of measurements required by any strategy for a guaranteed reconstruction, thus proving the optimality of their strategy.

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Cole and Yap have coined the term *geometric probing* for any measurement that gives simple geometric data, and initiated an active field of research on algorithmic approaches to robotic sensing problems. Other problems that have been investigated include the use of different types of probings [5], [7], [9], [13], [22], probing with uncertainty [5], [19], extensions to higher dimensions [5], using composite probes [13]–[15], and reconstructing nonconvex polyhedra [1]. Most of the above mentioned works addressed the same type of problem: the unknown object is *a priori* known to belong to a restricted class, such as polygons or polyhedra, a certain type of geometric probe is defined, and a reconstruction strategy is suggested and rigorously analyzed. The performance of the probing strategy is measured by the number of probings which guarantees exact reconstruction. This number is a function of the object complexity (V), and is usually compared with a proven lower bound on the number of probings required by any strategy.

In particular, it is worth mentioning the work of Li [13], who considered the reconstruction of convex polygon using line probings identical to the ones we investigate here. Li improved earlier results [5], [9] and suggested a probing strategy which guarantees complete reconstruction after no more than $3V + 1$ probings. He also derived a lower bound of $3V + 1$ on the number of measurements required by any strategy for a guaranteed reconstruction, thereby proving the optimality of his strategy.

Although the geometric probing approach discussed above provides a rigorous computational method for treating learning problems in robotics, it seems to miss an important point: the world is not polygonal, and basing the approach on its polygonality yields methods that are not applicable for real robotic tasks. For example, if the object is smooth in some small part of the boundary, then using the optimal strategies with line or finger probing will cause the information gathering process to concentrate on trying to get better locally while possibly failing to provide even approximate information on the rest of the boundary. This deficiency results from the insistence on perfect reconstruction. Such a demand does not have an engineering basis and obviously causes the reconstruction to be unnecessarily costly. Even if the object is a polygon with V vertices, reconstructing it exactly using line or finger probing requires about $3V$ measurements, which may be considered too high, particularly if V is large, and often an approximate reconstruction would suffice.

The approach discussed in this paper overcomes these deficiencies by considering the accuracy of the desired reconstruction as a parameter and by not restricting the objects considered to be polygonal. The exact reconstruction problem is modified into an approximation problem while maintaining the classical structure of geometric probing. An adaptive strategy for approximate reconstruction is sought for, and its figure of merit is defined as the number of probings it requires, in the worst case, in order to achieve a certain "certified approximate reconstruction." An upper bound on this figure of merit is proved in a rigorous way, and compared to a lower bound on the number of probings required by any strategy for achieving an approximation with the same precision.

The goal is to obtain enough information that would let us infer the shape of another object whose Hausdorff distance from the unknown one does not exceed a certain prespecified value. The Hausdorff distance was chosen since it seems to convey the intuitive notion of distance between two objects, and is also the most severe measure.

The two approaches to the reconstruction problem are analogous to the quantification of information, as done in information theory [17]. There, if the source of signals is discrete, the uncertainty in any signal can be measured by the Entropy function, and every signal can be stored (or transmitted) by a finite number of bits which may approach the entropy as close as we wish. On the other hand, if the source is a continuous one, then its Entropy is infinite, and every signal can be stored with a finite number of bits only when certain distortion is allowed, according to the Rate distortion theory [17] or the ε-entropy theory [24]. Analogously, in geometric probing problems, an object with a finite number of vertices (polygon), can be reconstructed precisely using a finite number of probings, while any non-polygonal object can be only reconstructed approximately with a finite number of probings. (One should not take this analogy to far. Note that the class of convex polygons is not discrete but a continuous one, and that the decrease in uncertainty caused by each probing is generally more than one bit.)

This probing problem may be looked upon as an approximation problem. Usually, one considers the situations where a given object (or function) is given, and another object (function) which approximates it, according to some metric, is sought for. If only partial information is known about the object, and an approximating object is desired then the distance between the object and any suggested approximation cannot be evaluated directly but rather inferred from the partial data and the assumptions on the object. This is similar to the situation where one looks for approximation to a function given only a discrete set of samples [20]. In fact, this is exactly the problem we have here as the probing results may be considered to be samples of the support function [6]. The question discussed in this paper is even more complicated as the partial data is collected dynamically and actively, and the strategy for collecting the data is to be considered too. This task may be called *blind approximation* as the approximation is inferred and evaluated without seeing (knowing) the object, and using only the partial data obtained through a sequence of probings.

The paper is organized as follows. We first look briefly on the problem of approximating convex shapes with polygons, a question which has attracted much attention in the mathematical literature. Before trying to suggest a blind approximation strategy, we start by showing a necessary and sufficient condition for being able to specify an approximation with a certain accuracy from probing results. Then, a lower bound on the number of probings required to specify the approximation is derived. In the next section a strategy is suggested and an upper bound on the number of probings it requires for proving a certain approximation follows. Finally, we discuss some open problems connected with this new direction in geometric probing and approximating theory.

II. Approximating Convex Sets with Polygons

The problem of approximating convex sets by polygons has attracted the interest of mathematicians for a long time. The problem may be approached using different metrics and different constraints on the approximating polygons. In this paper we restrict ourselves to the Hausdorff metric which defines the distance between two planar sets S and P, by

$$\delta^H(P,S) = \max\{\sup_{x\in P}\inf_{y\in S}|x-y|, \sup_{y\in S}\inf_{x\in P}|x-y|\} \quad (1)$$

where $|\cdot|$ is the Euclidean norm in R^2. This distance is the largest Euclidean distance between a point in one set and the closest point on the other.

Let P_n^c be the set of all convex polygons which contains polygons S whose number of vertices does not exceed n. The distance between a convex set S and P_n^c, depends on the metric δ and is defined by

$$\delta(S, P_n^c) = \inf_{Q\in P_n^c} \delta(S,Q). \quad (2)$$

For the Hausdorff metric δ^H, the following bound was derived by Popov [10]

$$\delta^H(S,P_n^c) \le L\frac{\tan(\frac{\pi}{n})}{2n} \text{ for } n = 3,4,\ldots \quad (3)$$

where L is the perimeter of S. Other researchers have shown properties of the best approximating polygons [12], [26] and

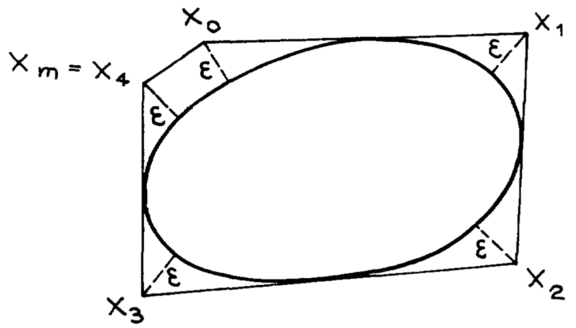

Fig. 2. Building external polygon to S, in distance ε.

have investigated the asymptotic properties of the approximation [8], [18].

In this paper we have chosen to use a less quoted result [3] which provides a simple constructive method for building an approximating external polygon to a convex set S, such that the following relation is guaranteed

$$n \leq \pi\left(\frac{L}{2\pi\varepsilon}+1\right)^{1/2}+1. \tag{4}$$

where ε is the approximation error.

To make the paper self-contained, the rest of this section is dedicated to presenting the proof of this result. It is based on the derivation of Bolour and Cover in their paper on the ε-entropy of the set of convex shapes included in the unit square [3].

The approximating polygon P is constructed as follows. Choose its first vertex x_0 as any point, external to S whose distance from its boundary is ε. Through this point, pass a tangent to S and extend it to the point x_1 whose distance from the boundary of S is also ε. Let x_1 be the successive vertex. Repeat this process, each step obtaining a new vertex x_i and a new side $x_{i-1}x_i$, as long as the curve $x_0, \cdots, x_i$ does not cut itself. To complete the approximating polygon, connect the last vertex x_m and x_0 by a line segment and let it be the last side (see Fig. 2). It is clear that the polygon $x_0, \cdots, x_m$ is external and convex, and that its Hausdorff distance from S is exactly ε. The following result is due to Bolour and Cover.

Theorem 1 [3]: Every convex set S with perimeter L can be approximated by an external convex polygon, built by the method described above, whose number of vertices, n is bounded by

$$n \leq \pi\left(\frac{L}{2\pi\varepsilon}+1\right)^{1/2}+1. \tag{5}$$

Proof: The approximating polygon $x_0, \cdots, x_m$, built by the above described method has $n = m+1$ vertices. Consider the tangency point of the ith ($i = 1, 2, \cdots, m$) side of the polygon and the angles defined by this side, as shown in Fig. 3. It is clear that

$$l_{i1} \geq \frac{\varepsilon}{\sin\alpha_{i1}}, \quad l_{i2} \geq \frac{\varepsilon}{\sin\alpha_{i2}}. \tag{6}$$

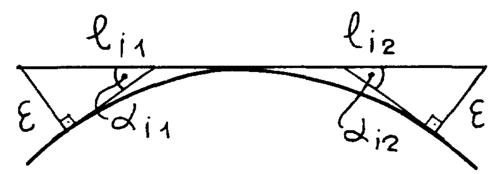

Fig. 3. One side of the external polygon.

Another relation

$$\sum_{i=1}^{m} \alpha_{i1}+\alpha_{i2} \leq 2\pi \tag{7}$$

follows since all the angles α_{i1}, α_{i2} are part of the external angles of the same convex polygon. Finally, the relation

$$\sum_{i=1}^{m}(l_{i1}+l_{i2}) \leq L+2\pi\varepsilon \tag{8}$$

holds since the approximating polygon is trivially included in the set $S \oplus 0_\varepsilon$ whose perimeter is $L + 2\pi\varepsilon$ (where 0_ε is a circle of radius ε and $\oplus$ denotes direct sum operation) [25]. From these relations and from the convexity (cup) of the $1/\sin$ function, it follows that

$$L+2\pi\varepsilon \geq \sum_{i=1}^{m}(l_{i1}+l_{i2}) \geq \varepsilon\sum_{i=1}^{m}\left(\frac{1}{\sin\alpha_{i1}}+\frac{1}{\sin\alpha_{i2}}\right) \tag{9}$$

$$\geq 2m\varepsilon\frac{1}{\sin\sum_{i=1}^{m}\frac{\alpha_{i1}+\alpha_{i2}}{2m}} \geq 2m\varepsilon\frac{1}{\sin\frac{\pi}{m}} \geq 2m\varepsilon\frac{1}{\frac{\pi}{m}} \geq 2m^2\frac{\varepsilon}{\pi} \tag{10}$$

and thus

$$m \leq \pi\left(\frac{L}{2\pi\varepsilon}+1\right)^{1/2}. \tag{11}$$

and the theorem follows.

Asymptotically, this result coincides with the result derived by Popov [10] and by McClure and Vitale [18]. It will be used later in this paper.

III. A Formal Definition of the Approximate Reconstruction Problem

Let S be an unknown planar convex set on which data may be obtained only as a result of the following discrete measurement process. Before each measurement, a direction (denoted by the unit vector B_j) may be arbitrarily specified. The result of the the measurement is a positive number ρ_j satisfying the following relations.

$$\exists\bar{x} \in S \text{ s.t. } \quad \bar{B}_j \cdot \bar{x} - \rho_j = 0 \tag{12}$$

$$\forall\bar{x} \in S \quad \bar{B}_j \cdot \bar{x} - \rho_j \leq 0. \tag{13}$$

The clear geometrical interpretation is that each measurement is done by choosing a direction on the plane, and moving a line perpendicular to this direction, from infinity until it touches the object at distance ρ_j from the origin. Thus, each such measurement, denoted line probing, which may be described by the pair (B_j, ρ_j) reveals a tangent line to the object (see Fig. 1). After some measurements are done, inference about the object shape may be carried. In particular, we are interested in the case that the data gathered by the measurements enable one to specify a set S' whose Hausdorff distance from the unknown set is guaranteed to be ε or less. Such a set S' is called *a certified approximation* and is denoted CA_ε. If the certified approximation CA_ε is a polygon, then it is denoted CAP_ε.

Define a probing strategy to be a rule for choosing the direction B_j, that may depend adaptively on all previous probing results. A strategy is considered to be a better one if it requires, in the worst case, a smaller number of probings for achieving a certified approximation with a certain precision.

First, let us make clear how one can infer a certified approximation from the results of the probing process described above. The following notation will be very useful throughout the paper.

Given a sequence of j measurements (B_i, ρ_i), $i = 1, 2, \cdots, j$, the set R_j, which reveals nearly all the data obtained in the probing process may be defined

$$R_j = \{\bar{x} \mid \bar{B}_i \cdot \bar{x} \leq \rho_i, \quad i = 1, 2, \cdots, j\}. \tag{14}$$

Note that the unknown object S is included in R_j, and that the set R_j is included in R_{j-1} and gets smaller with increasing number of measurements. It would be convenient if the set R_j describes all the first j measurements. However, if the object has corners, then it is possible that three of the tangent lines obtained by the probings, intersect in the same point, and thus one of them is not a side of R_j, and is not revealed when only the set R_j is known. This situation would make the description on the probing results and the derivation of the results in this paper much more cumbersome.

To care for this inconvenience, we shall make a little deviation from the previous formal definition of the probing process and assume, only for notation purposes, that the object probed is not S but another object S_r, identical to S except that all its corners are *rounded* to small infinite curvature arcs. Now, the set R_j describes all the results of the first j measurements, since each measurement yields a tangent line which corresponds to one of its sides. The difference between S and S_r is a collection of corners and their infinitesimal neiborhood, and any approximation to the set S_r, is also close to these corners, and approximates S. Therefore the difference between S and S_r is ignored.

In the following lines we will provide a necessary and sufficient condition on the ability to infer a certified approximation CA_ε from the set R_j. We shall need the following definitions.

Definition: Let P be a convex polygon with M vertices $v_1, \cdots, v_m$. Define h_i^*, the generalized height of the vertex v_i, as the distance between v_i and the closest point on the line segment $v_{i-1}v_{i+1}$.

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Note: Clearly, the generalized height of a vertex is never smaller than its height.

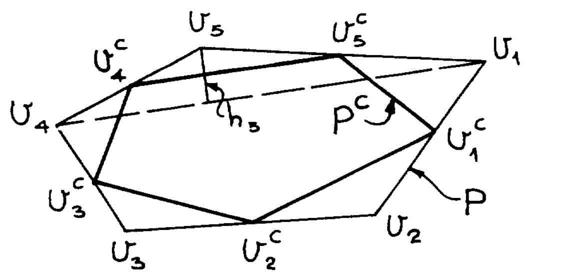

Fig. 4. A convex polygon P and its central polygon P^c.

Definition: Let P be a convex polygon with M vertices $v_1, \cdots, v_m$. We shall say that the (generalized) height of the polygon is h if $h_i \leq h$ ($h_i^* \leq h$) for all i.

Definition: Let P be a convex polygon with M vertices $v_1, \cdots, v_m$. Define its central polygon P^c, as the (convex) polygon whose vertices $v_1^c, \cdots, v_m^c$ are the centers of the sides of P. See Fig. 4, in which a polygon P and its central polygon P^c are described.

Note that if the true set R_j includes a vertex in which three tangent lines intersect, then, by our assumptions, we treat this vertex as two infinitesimally close vertices whose heights (and generalized heights) are also infinitesimally small.

Using these notations, the following condition can be proved.

Theorem 2: Let R_j be a polygon created as the result of the probing process after j probings. Then R_j being of generalized height 2ε is a necessary and sufficient condition to inferring a certified approximation CA_ε to the unknown object S. If the condition holds, then the Hausdorff distance of R_j^c, the central polygon of R_j, from the unknown object S does not exceed ε, i.e., R_j^c is a CAP_ε.

Proof: The proof is simple and geometric in nature. First, we prove the sufficiency of the condition. Clearly, if the generalized height of R_j is 2ε, then its height is not higher. Denote by $t_1, \cdots, t_j$ the points in which S is tangent to the sides of R_j. (The point t_i is chosen arbitrarily as one of the points of S which are included in the side v_iv_{i+1} of R_j, see Fig. 5.) Clearly, the distance of every point that lies inside one of the triangles $v_{i-1}v_iv_{i+1}$ from the boundary of R_j^c is ε or less. Every point of the boundary of S lies inside R_j (or on its boundary) but outside the polygon $t_1, t_2, \cdots, t_j$ (or on its boundary), thus it must lie inside one of the triangles $v_{i-1}v_iv_{i+1}$. It follows that for every point on the boundary of S, there is at least one point on the boundary of R_j^c whose distance is ε or less.

$$\sup_{x \in S} \inf_{y \in R_j^c} |x - y| \leq \varepsilon. \tag{15}$$

Consider now an arbitrary point on the side $v_{i-1}^c v_i^c$ of R_j^c. We shall show that there is a point in S, whose distance is less than ε. Define a rectangle $abcd$ with side parallel to $v_{i-1}^c v_i^c$, as shown in Fig. 5. The point considered may be included in one of the segments $v_{i-1}^c e$, ef, and fv_i^c specified on $v_{i-1}^c v_i^c$ by the lines v_iv_{i-2} and v_iv_{i+2}. If the point lies inside the segment fv_i^c, consider two cases. In the first one, where the tangency point t_i lies in $v_i^c v_{i+1}$, the segment $t_{i-1}t_i$ intersects

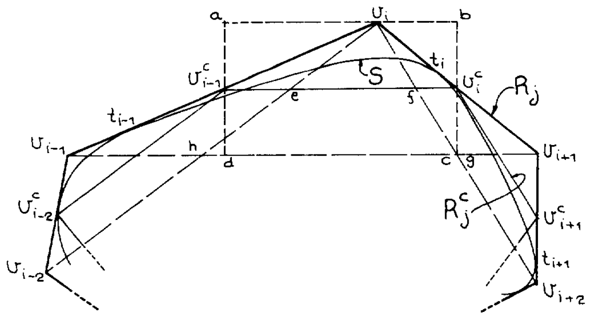

Fig. 5. A part of the boundaries of S. R_j and R_j^c.

with the segment bc, and thus every point inside fv_i^c has a point in $t_{i-1}t_i$ between itself and its nearest point on ab, or on cd, i.e., there is point on S whose distance is ε or less. In the second case, the tangency point lies in $v_iv_i^c$. Then, for each point in fv_i^c, there is a point either in $t_{i-1}t_i$ or in t_it_{i+1} between itself and its nearest point on one of the lines ab or cd. It follows that there is a point on S whose distance is ε or smaller. (This is the case described in Fig. 5.) Points that lie inside the segments $v_{i-1}^c e$ or ef are considered by similar argumentation, and thus

$$\sup_{y \in R_j^c} \inf_{x \in S} |x - y| \leq \varepsilon. \tag{16}$$

From the two inequalities proved, it follows that

$$\delta^H(S, R_j^c) \leq \varepsilon \tag{17}$$

and thus, if the generalized height of R_j is 2ε or less, then it is a sufficient condition for R_j^c being a CAP_ε.

The necessity of the condition is derived by assuming that there is a certified approximation and showing that the generalized height of the R_j cannot be too high. Suppose $h_i^* > 2\varepsilon$ and consider a circle of radius ε around the vertex v_i. If the approximating set S' does not have a point inside this circle, then its distance from v_i is greater than ε. The unknown object S may include the point v_i and thus S' is not a CA_ε of S. If the approximating set S' has a point inside the circle, then the distance of this point from the segment $v_{i-1}v_{i+1}$ is greater than ε. Since the segment $v_{i-1}v_i$ may be a part of the boundary of the unknown object S, it follows that S' is not a CA_ε of S.

The theorem just proved provides us with a stopping rule, and a direct method for finding the required certified approximation from the given probing results. The notion of the generalized height is essential for providing the general sufficient condition for the existence of a certified approximation after arbitrary probing process. The angles at the vertices of R_j created by the strategy proposed in this paper, are never smaller than $\frac{\pi}{2}$, implying that the generalized height and the (simple) height are identical. Therefore, our stopping rule will examine the height of R_j. Before a strategy is proposed, let us first investigate the performance achievable from any strategy by deriving a lower bound of the number of probings required for achieving a certified approximation.

IV. A Lower Bound on the Number of Probings Required for Achieving a Certified Approximation with Precision ε

Consider the following problem. A convex set S is given and a polygon P with the following properties is sought for. P should include S with every side of it being tangent to the boundary of S, its generalized height (as defined before) must not exceed the number 2ε and its number of vertices should be minimal. Such a polygon, from which the certified approximation to S (P^c) may be clearly inferred, could be thought as a result of an optimal probing strategy which gets correct information about S and needs only to produce a certified approximation. The number of sides of this polygon is a lower bound on the number of probings required to find a certified approximation using the best possible strategy, as the additional information about the shape of S, cannot deteriorate the performance of the probing strategy.

Since we are looking for an optimal strategy with respect to the worst case, it follows that in order to show a limitation of any strategy, it suffices to treat a single case. Thus, we develop the lower bound on the number of sides of P for the case that S is a circle with radius R.

Let m be a smallest number for which a regular polygon P_m, with m sides tangent to the circle, has a height of 2ε or less. The corresponding regular polygon P_{m-1} with $m-1$ sides has a height greater than 2ε and so is any other polygon with $m-1$ sides whose sides are tangent to the circle. Note that for regular polygons, the height and the generalized height are the same. Consider a part of the P_m polygon described in Fig. 6. Simple trigonometry yields that

$$h = 2R \sin\theta \tan\theta \tag{18}$$

and if the relations

$$h \leq 2\varepsilon\; \theta = \frac{2\pi}{2m} \tag{19}$$

and

$$\sin\theta \geq \frac{2}{\pi}\theta, \quad \tan\theta \geq \frac{2}{\pi}\theta \tag{20}$$

are used then we get the following lower bound on the number of sides, m

$$m \geq 2\left(\frac{L}{2\pi\varepsilon}\right)^{1/2} \triangleq \mathrm{LB}. \tag{21}$$

As argued before, the bound LB holds also for the number of probings required to find a certified approximation with a given precision ε using any possible strategy.

For small values of ε/L we may get the following tighter asymptotic bound inserting θ as the limit of $\sin\theta$ and $\tan\theta$.

$$m \geq \pi\left(\frac{L}{2\pi\varepsilon}\right)^{1/2}. \tag{22}$$

These results will be used as reference to the performance of the strategy developed in the next section.

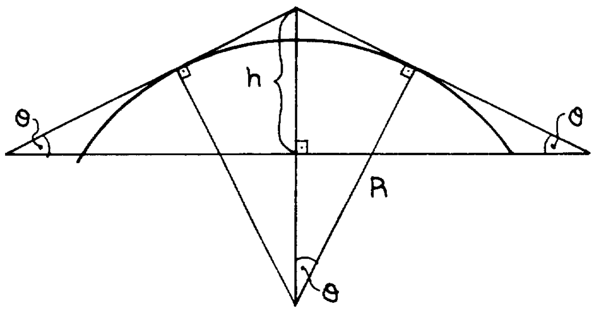

Fig. 6. A part of a regular polygon that circumscribes a circle.

V. The Probing Strategy

The proposed probing strategy is based on the following principles: The probing is done in stages such that, at each stage a unique probing is done for every vertex of R_j created in the previous stage, whose height exceeds 2ε. This probing deletes the corresponding vertex of R_j and creates two adjacent vertices instead of it. First, all the directions are sampled uniformly and coarsely, but after the height of some vertices becomes smaller than the threshold 2ε, no further probing is done on them and all probing effort is concentrated in the places where the uncertainty is still higher than allowed. This hierarchical and adaptive nature allows the strategy to use much less probings than a strategy based on a uniform sampling method.

In the rest of this section the strategy will be presented formally, and then in the following sections its performance will be evaluated.

We shall use the following notations: Every probing is associated with a direction vector B_j or a tangent angle θ_j. Probings with identical directions reveal the same information, and thus it is assumed that they are not done. It follows that a unique vertex v_j of R_{j-1} may be associated to the jth probing. The height h_j is defined as the height of v_j. The set S_k is a uniform sampling of the $[0, 2\pi]$ range, which gets finer with increasing parameter k

$$S_k = \{i \cdot \frac{2\pi}{2^k} \mid i = 1, 2, 3, \cdots, 2^k\} \tag{23}$$

The set T_k is defined by

$$T_k = \{x \mid x \in S_k, x \notin \bigcup_{i=1}^{k-1} S_i\}. \tag{24}$$

Both are useful in defining the probing strategy.

The Probing Strategy
probe until the height of all vertices of R_j is 2ε or less.
start from $k = 1$ and increase k in each stage
stage k
probe sequentially in all angles satisfying
a. $\theta_j \in T_k$
b. $h_j > 2\varepsilon$

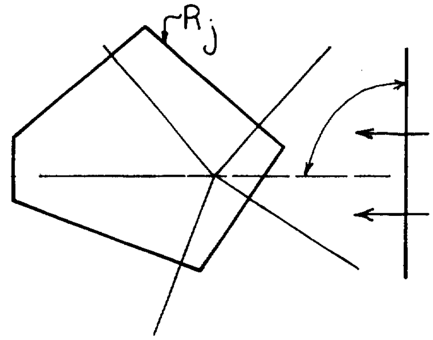

Fig. 7. A star diagram describing the probing process.

Note that each stage usually consists of more than one probing. The set R_j and the value of $h_i : i = 1, 2, \cdots, j$ are determined after each probing (and not only after each stage). The strategy depends on the parameter 2ε and the probing process terminates only after the height of the polygon R_j is 2ε or less.

Some insight can be gained by building a star diagram to represent the probing strategy. In this diagram, described in Fig. 7, every probing is represented by a ray whose direction is perpendicular to the corresponding side of R_j. Each sector in the diagram corresponds to a unique vertex of R_j. Some simple characteristics of the probing results can be inferred from this diagram. Since the set T_k contains one angle between any two adjacent angles in $\cup_{i=1}^{k-1} T_i$, it follows that, in the kth stage, there is a potential for making at least one probing in each of the sectors created by the probings done in the previous stages. Each of these probings is done if the height of the corresponding vertex is higher than the threshold 2ε. However, as the height of the vertex cannot increase due to probings done on adjacent vertices, it follows that if in some stage k, no probing is done in a certain sector, then no probing will be done in this sector in all subsequent stages. It further follows that any probing done in the kth stage is done inside a sector created in the $(k-1)$-stage, it is the only probing done in this sector and it divides the sector into two equal $2\pi \cdot 2^{-k}$ parts. In other words, each new side added to R_j by a probing done in the kth stage makes a $2\pi \cdot 2^{-k}$ angle with its adjacent sides.

Note that the angles at the vertices of R_j created by the proposed strategy, are never smaller than $\frac{\pi}{2}$, implying that the generalized height and the (simple) height are identical.

VI. An Upper Bound on the Number of Probings Required to Find a Certified Approximation to a Convex Polygon

In this section we make an initial step in evaluating the proposed probing strategy, and derive an upper bound on the number of probings required for achieving a certified approximation with precision ε (CA_ε) to a convex polygon. The main reason for investigating this question is that it would serve as a preliminary step towards finding such a bound for general convex objects. One may also compare the results

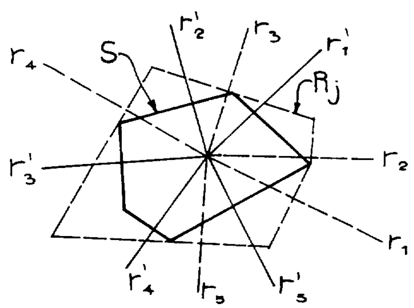

Fig. 8. A star diagram describing the probing process, the unknown object and the types of the vertices.

predicted by this bound to the results of the optimal probing strategy for polygons, already derived by Li, which guarantees exact reconstruction after no more than $3V+1$ probings, where V denotes the number of vertices of the unknown polygon [13]. Even if it may be assumed that the object is polygonal, a faster approximate reconstruction may be advantageous to the slower exact reconstruction suggested by the traditional method.

The star diagram, introduced in the previous section, is used for describing and characterizing the probing process. The sides of the unknown polygon are represented by a new set of V rays, perpendicular to them. In contrast to the rays defined before, which correspond to the probings and are denoted by r_i (Fig. 8), the new rays are denoted by r_i'.

In the next lines we adopt the following terminology. The word *sector* is used only for sectors between rays corresponding to probings, and the word *ray* is used only for rays corresponding to sides of the unknown object.

It will be useful to divide the vertices of R_j and their corresponding sectors into three classes:

(i) Vertices that are also vertices of the unknown set S. Such vertices are created from two probings which meet the same vertex of S and correspond to sectors which do not contain any rays (e.g., the sector $r_1 r_2$ in Fig. 8).

(ii) Vertices created from two probings which meet adjacent vertices of S, and correspond to sectors which contain exactly one ray (e.g., the sector $r_2 r_3$ in Fig. 8).

(iii) The rest of R_j's vertices, each created from two probings which meet different vertices of S which are not adjacent. These vertices correspond to sectors which contain more than one ray each (e.g., the sector $r_4 r_5$ in Fig. 8).

Note that R_j cannot have more than V vertices which are type (ii) or type (iii). The classification to classes is done only for the analysis, and there is, generally, no way to distinguish between vertices of different classes when only the set R_j is given. The following derivation of a bound on the number of probings required to find a certified approximation to a polygon is based on this classification.

Theorem 3: Let S be a convex polygon with V vertices and perimeter L. Assume $V \leq \frac{16L}{\varepsilon}$. Then, probing according to the proposed strategy guarantees that after $j = V \log_2\left(\frac{32L}{\varepsilon V}\right)$ probings, the height of R_j cannot exceed 2ε and that R_j^c is a certified approximation to S.

Proof: Each probing bisects a certain sector of the star diagram and corresponds to a certain vertex of R_j. The notation *the probing is done on a vertex* denotes this correspondence. The probings are counted according to the classification of the corresponding vertices described above. First only probings which are done on type (i) vertices are considered. Let v be a type (i) vertex of R_j. Such a vertex corresponds to a vertex of S, and if its height is greater than 2ε then it is probed. After the probing, the vertex v is deleted and two vertices v_1 and v_2 which are both type (i) and correspond to the same vertex of S, replace it. (This is the probing result according to the assumption made, that all vertices of S are substituted by small infinitesimal circular arcs. This assumption simplifies the derivation since it allows R_j to be a full representation of the first j probings.) No further probing will be done on these new vertices, as their heights are infinitesimally small (and certainly lower than 2ε). Thus, for each vertex of the set S, at most one probing is done on the corresponding type (i) vertices, and no more than V probings are done on type (i) vertices.

Consider now a probing done on a type (ii) vertex. The corresponding sector contains exactly one ray, and a probing done exactly in this direction yields a tangent line which coincides with a side of S, deletes the type (ii) vertex and replaces it by two type (i) vertices. Probing in any other direction in the sector yields a tangent line which passes through a vertex of S, deletes the old type (ii) vertex and replaces it by one type (i) vertex and one type (ii) vertex. The corresponding sector is halved into two equal sectors of $2\pi \cdot 2^{-k}$ angle, where k is the stage in which the probing is done. Let l_i be the length of the chord $v_{i-1}v_{i+1}$ in R_j S (see Fig. 9(a)). The angle between the two tangent lines forming the new type (ii) vertex is $2\pi \cdot 2^{-k}$, implying that the maximal height of the new vertex v_i is achieved if $v_{i-1}v_i = v_i v_{i+1}$, and is bounded by

$$h_i \leq \frac{l_i}{2} \tan \frac{2\pi}{2^{k+1}} \leq \frac{l_i}{2} \cdot \frac{2\pi}{2^{k+1}} \cdot \frac{4}{\pi} = \frac{l_i}{2^{k-1}} \tag{25}$$

($k \geq 2$). As at most one probing is done in each stage on the type (ii) vertex which corresponds to this side, it follows that $h_i \leq 2\varepsilon$ after no more than n_i probings done on this vertex, where

$$n_i \leq \lceil \log_2 \frac{l_i}{\varepsilon} \rceil. \tag{26}$$

(The type (ii) vertex that corresponds to a certain side of S, changes its location, of course, after each probing aimed at it, but it does not split into two vertices and thus we still refer it to as the same vertex.)

Consider now a probing done on a type (iii) vertex. One possible result may be that the vertex is replaced by a type (i) vertex and by another type (iii) vertex. If all the results are of this kind then it is guaranteed that the height of the vertex is 2ε or less after n_i probings, where

$$n_i \leq \lceil \log_2 \frac{d_i}{\varepsilon} \rceil. \tag{27}$$

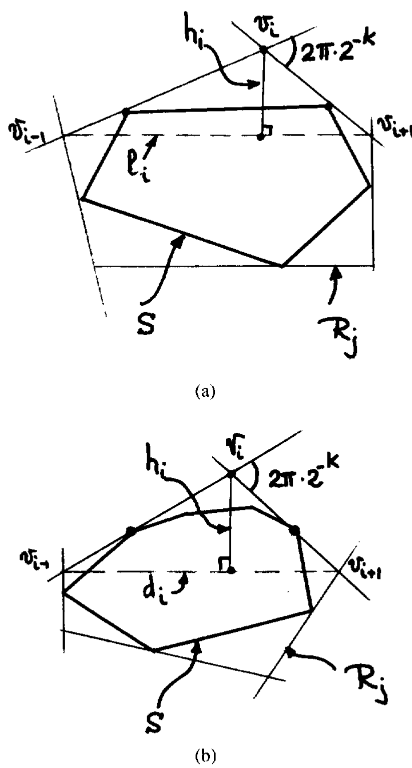

Fig. 9. Probing a type (ii) vertex and a type (iii) vertex.

(d_i is the length of the chord $v_{i-1}v_{i+1}$ in R_j S. See Fig. 9b). It might be, however, that the probing result is that the type (iii) vertex is deleted and two type (ii) or type (iii) vertices replace it. If the height of both vertices is 2ε or less, then the above argument and the above bound (27) hold. If the height of at least one of these vertices is higher than 2ε, then the probings done on the original type (iii) vertex, are counted as done on this new vertex. If the probing is terminated when a type (ii) vertex is probed and two vertices with heights 2ε or smaller are created, then applying this argument recursively implies that the number of probings done on this vertex (together with the probings done on its ancestors) cannot exceed the bound (26), where l_i is the length of the corresponding side. Similarly, if the probing is terminated when a type (iii) is probed, then the bound (27) holds, where d_i is the length of the corresponding chord.

The number of probings required to reduce the height of all vertices to 2ε or less may be bounded by summing all contributions

$$n \leq \sum_{(i)} 1 + \sum_{(ii)} \lceil log_2 \frac{l_i}{\varepsilon} \rceil + \sum_{(iii)} \lceil log_2 \frac{d_i}{\varepsilon} \rceil. \tag{28}$$

The probings are divided into the three sums. The first sum includes all the probings done on type (i) vertices. Every term of the second sum includes one probing done on a type (ii) vertex which results in two vertices with height 2ε or less, and all the probings which lead to this probing, which may be type (ii) or type (iii). Every term in the third sum includes one probing done on a type (iii) vertex which result in two vertices with height 2ε or less, and all the probings which lead to this probing, which are also type (iii). It is possible that this bound may be tightened as many probings are counted twice.

Suppose that V' probings were done on type (i) vertices and V'' probings were done on type (ii) and type (iii) vertices. $(V', V'' \leq V)$. Each of these chords corresponds to a certain vertex v_i of R_j, and a certain height, smaller than 2ε, associated with it.

A subtle point to observe is that, the number of probings required to reduce the height h_i below 2ε, as given by (26) corresponds to a chord length l_i at that stage, which may be larger than the corresponding chord $l*_i$ in the final R_j, from which the certified approximation is infered. It is simple to show however, that the original chord length l_i cannot be more than twice as large as the final chord length l_i^*.

Note now that the chords of the final R_j may be grouped into two subsets of alternating chords, which form two convex polygons (with possibly one missing side). These convex polygons are included in R_4, the perimeter of which is bounded by $2L$. Therefore,

$$\sum_{(ii)} l_i + \sum_{(iii)} d_i \leq 8L. \tag{29}$$

By using this relation and the convexity (cap) of the logarithmic function, it follows that by using the proposed strategy, no more than n probings are required to lower the height of every vertex in R_j to 2ε or less, and

$$\begin{aligned} n &\leq V' + \sum_{(ii)} \log_2 \left(\frac{2l_i}{\varepsilon}\right) + \sum_{(iii)} \log_2 \left(\frac{2d_i}{\varepsilon}\right) \\ &\leq V' + V'' \log_2 \left(\frac{16L}{\varepsilon V''}\right) \leq V + V \log_2 \left(\frac{16L}{\varepsilon V}\right) \\ &= V \log_2 \left(\frac{32L}{\varepsilon V}\right) \end{aligned} \tag{30}$$

(The function $V'' \log_2 \left(\frac{16L}{\varepsilon V''}\right)$ is increasing in V'' for the specified range and thus the last inequality is justified.)

Hence, after no more than $V \log_2 \left(\frac{32L}{\varepsilon V}\right)$ probings, R_j^c is a CAP_ε.

The derivation of the bound on the number of probings, done in the above theorem, depends on V', V'', and ε, with no interdependence between them. Hence, a polygon with many vertices may require a lot of probings for achieving the certified approximation. This is not the case, however, and the bound may be tightened by arguing that the number of vertices and sides found by the strategy (i.e., V' and V'') cannot be arbitrarily big and independent of the relation between ε and L. Our main objective in deriving the above bound is not to predict the performance of the probing strategy on polygonal objects but rather to use it for developing a bound for the general convex object, hence we shall not elaborate on this subject.

VII. Probing with a Positive Error

In this section we perform a further step towards deriving the bound in the general case. A special kind of a line probing, denoted *consistent positive error line probing* is defined.

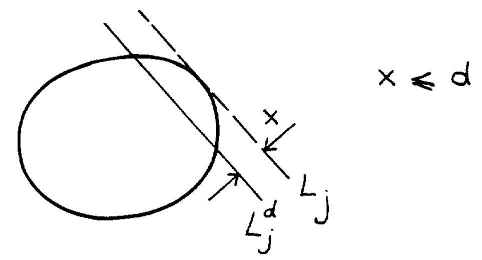

Fig. 10. Probing with positive error.

Probing with such probes according to the proposed probing strategy results in a polygon R_j, and the height of this set can be predicted in a way similar to the one used for usual line probing in the last section. In the next section, it is shown how to use this result to derive the general bound.

Recall that a usual line probing is specified by a direction B_j and its result is a positive number ρ_j^0 defining a line

$$L_j = \{\bar{x} \mid \bar{B}_j \cdot \bar{x} - \rho_j^0 = 0\}. \tag{31}$$

For the same direction B_j, the positive error line probing gives the result ρ_j^d satisfying

$$0 \le \rho_j^0 - \rho_j^d \le d. \tag{32}$$

Intuitively, the pair (B_j, ρ_j^d) defines a line probing which *penetrates into the object* up to a maximal depth d (see (Fig. 10). A consistent positive error line probing also guarantees that all probings create a consistent set of probings, i.e., there is a convex object which satisfies the relations (12) and (13) for all probings. It is not difficult to see that a sufficient and necessary condition for consistency is

$$L_i^d \cap R_j^d \ne \emptyset \quad i = 1, 2, \cdots, j. \tag{33}$$

R_j^d is defined similarly to R_j and the superscript d denotes that it results from line probings with positive error. We leave the immediate question, how can one guarantee the consistency, without an answer. The objective for defining this kind of probing is not to serve as a model for some real measurement process but rather to be used as a mathematical tool in a situation where the consistency will be self evident. A bound on the number of probings required to lower the height of R_j^d to 2ε or less is derived in the following lines.

Lemma 1: Let S be a convex polygon with V vertices and perimeter L. Assume $V \le \frac{64L}{\varepsilon}$. Then, probing according to the proposed strategy with consistent positive error line probes associated with error $d = \varepsilon$ guarantees that after no more than $j = V \log_2\left(\frac{128L}{\varepsilon V}\right)$ probings, the height of R_j^d is 2ε or less.

Proof: As done in the previous proof, the probings are classified and counted according to the classification of their corresponding vertices to type (i), (ii) and (iii). Consider the probing done on a type (i) vertex of R_j^d. The probing deletes the vertex and replaces it with two new vertices. The height h_i of such a new vertex v_i increases with the distance of the adjacent old vertex v_{i+1}, and is maximal when v_{i+1} is in infinity (see Fig. 11(a)). Then

$$h_{i\,\max} = v_{i-1}v_i \sin\theta_1 \tag{34}$$

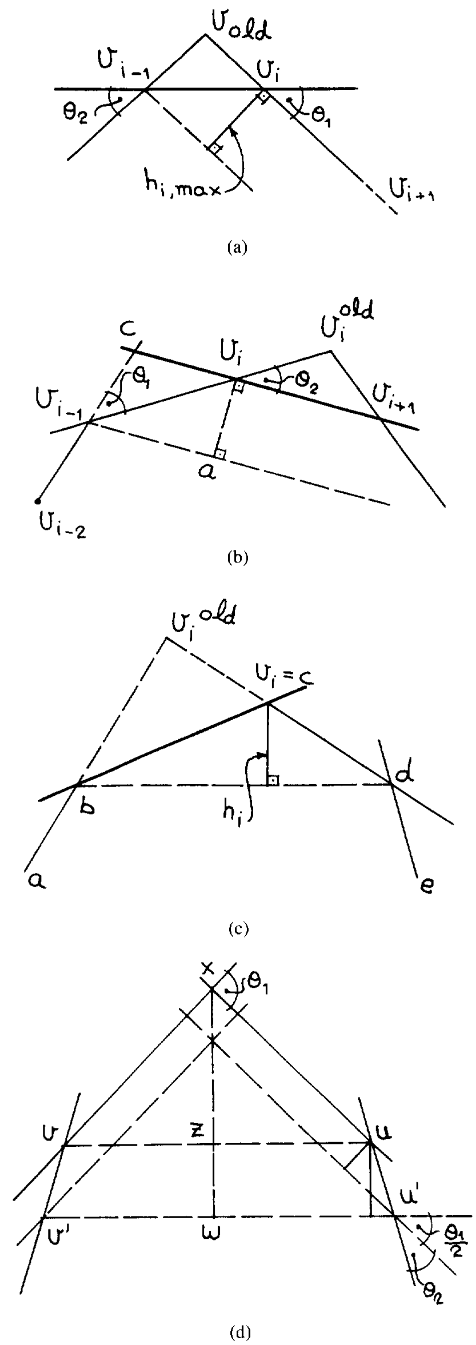

Fig. 11. Various situations of probing a polygon with the positive error probe (see the proof of the lemma).

where the angle θ_1 is $2\pi \cdot 2^{-k}$. h_i takes the worst case value when the segment $v_{i-1}v_i$ is maximal, or when the tangent line $v_{i-1}v_i$ penetrates to the maximal depth d. In this case

$$h_{i\,\max} = 2d \cdot \cot\theta_1 \cdot \sin\theta_1 = 2d\cos\theta_1 < 2d. \tag{35}$$

For exact probing ($d = 0$), it is guaranteed that the height of vertices created when a type (i) vertex is probed is infinitesimally close to zero, while for positive error probing, we have been able to show only that the height of these vertices is upper bounded: $h_i < 2d$. However, if we choose $\varepsilon = d$, this is enough to guarantee that for each vertex of the set S, at most one probing is done on the corresponding type (i) vertices, and no more than V probings are done on all the type (i) vertices.

Consider now a probing done on a type (ii) vertex of R_j. The probing deletes the vertex and replaces it with two new vertices: one of type (i) and the other of type (ii). Consider first the resulting type (i) vertex and assume that it corresponds to the vertex v_s of S (see Fig. 11b). If R_j already includes a vertex (v_{i-1}) which corresponds to the vertex v_s of S, then the height of the new type (i) vertex cannot be more than $2d$, and if $\varepsilon = d$, no further probings are done on it. This result is proved as follows. Denote by a the projection of the new vertex v_i on a line parallel to the probe and passing through the old vertex v_{i-1}. The length $v_i a$ is an upper bound to the maximal height of the vertex v_i, achieved only when the vertex v_{i+1} is at infinite distance (see Fig. 11(b)), hence

$$\begin{aligned} h_i \leq v_i a &= v_i v_{i-1} \sin\theta_2 \leq (d\cot\theta_1 + d\cot\theta_2)\sin\theta_2 \\ &= d(\cos\theta_1 \cdot \frac{\sin\theta_2}{\sin\theta_1} + \cos\theta_2) \leq 2d. \end{aligned} \tag{36}$$

Note that Fig. 11(b) describes the worst case situation ($v_{i-1}v_i$ maximal) where two probings result in tangent lines $v_{i-1}v_s$ and $v_i v_s$ passing through v_s, and one probing results in the tangent line $v_{i-1}v_i$, which penetrates to the maximal depth d. The last inequality follows since the angle θ_1 is created in the same stage k as the angle θ_2 or earlier, and thus it cannot be smaller. (Recall that an angle of R_j created at the kth stage has always a value of $2\pi \cdot 2^{-k}$.)

Consider now the new type (ii) vertex resulted from the probing. This vertex denoted v_i in Fig. 11(c), is created in the intersection between the tangent lines $v_{i-1}v_i$ and $v_i v_{i+1}$, where $v_{i-1}v_i$ denotes the tangent line revealed by the current probing, and $v_i v_{i+1}$ is a previous one (see Fig. 11(c)). For the worst case analysis, the results of the probing are chosen such that the vertices v_{i-1}, v_i, and v_{i+1} are in a position that brings the height to a maximum. Choosing the probings' results is done for each vertex separately, and is not based on the dependencies between them, and thus achieve a value which is worse (bigger) than the real worst case value. Suppose the vertex v_i is in some fixed place, and then choose the results of the probing to maximize the distance between v_i and the line $v_{i-1}v_{i+1}$ This will be the case if the tangent lines $v_{i-2}v_{i-1}$ and $v_{i+1}v_{i+2}$ are revealed by probings which make no error, and the tangent lines $v_{i-1}v_i$ and $v_i v_{i+1}$ are revealed by probings which penetrate to the maximal depth. Suppose now that the line $v_{i-1}v_{i+1}$ is in some fixed place and then choose the results of the probing to maximize the distance between $v_{i-1}v_{i+1}$ and the vertex v_i. This will be the case if the tangent lines $v_{i-1}v_i$ and $v_i v_{i+1}$ are revealed by probings which make no error. For a given angle between the lines $v_{i-1}v_i$ and $v_i v_{i+1}$, the maximal height is obtained when $v_{i-1}v_i = v_i v_{i+1}$ (see Fig. 11(d)) where x, u, and v are the vertices obtained from exact probing, and x, u', and v' are the vertices in the worst case discussed above). In the worst case the height h_i is the sum of the height we would get by exact probing height $h_{\text{no error}} = xz$ and the length wz caused by the probing error.

$$\begin{aligned} h_{\text{worst case}} &= h_{\text{noerror}} + wz \\ &= h_{\text{no error}} + \frac{d}{\sin\theta_2}\sin\left(\frac{\theta_1}{2} + \theta_2\right). \end{aligned} \tag{37}$$

(see Fig. 11(d)). θ_1 is the angle of the new sector created by the probing and is equal to the new external angle of R_j, implying that it cannot be bigger than θ_2 which is a similarly created in a previous probing. Therefore, $h_{\text{worst case}}$ achieves a maximal value if θ_1 is equal to $\theta_2 (= \theta)$.

$$h_{\text{worst case}} = h_{\text{no error}} + d\frac{\frac{\sin 3\theta}{2}}{\sin\theta} \leq h_{\text{no error}} + \frac{3d}{2} \tag{38}$$

Suppose it is possible to guarantee that the height of all vertices would be 0.5ε or less if the probing is done without error, i.e., $h_{\text{no error}} \leq 0.5\varepsilon$. Then, if positive error probing with $d = \varepsilon$ is done, it follows that the actual height of all vertices cannot exceed 2ε. As proved in Theorem 3, this condition is guaranteed after no more than n_i probings done on this vertex or its ancestors, where

$$n_i \leq \lceil \log_2\left(\frac{4l_i}{\varepsilon}\right)\rceil \log_2\left(\frac{8l_i}{\varepsilon}\right). \tag{39}$$

For probing which are done on type (iii) vertices the argumentation is similar, and thus the number of probings required to lower the height of a type (iii) vertex to 2ε or less is bounded by

$$n_i \leq \lceil \log_2\left(\frac{4d_i}{\varepsilon}\right)\rceil \leq \log_2\left(\frac{8d_i}{\varepsilon}\right). \tag{40}$$

Summarizing all contributions and using the relation

$$\sum_{(ii)} l_i + \sum_{(iii)} d_i \leq 8L \tag{41}$$

and the convexity of the logarithmic function, it follows that the total number of probings n required to lower the height of R_j^d to 2ε or less, is bounded by

$$n \leq V + V\log_2\left(\frac{64L}{\varepsilon V}\right) = V\log_2\left(\frac{128L}{\varepsilon V}\right) \tag{42}$$

and the Lemma is proved.

The sides of the set R_j^d, created by the probing process, are not tangent to S, and the set R_j^d itself does not necessarily include S. Hence, R_j^d being of height 2ε does not imply that its central polygon is a CA_ε.

VIII. An Upper Bound on the Number of Probings Required to Find a Certified Approximation to a Convex Object

In this section it is shown that no more then $0\left(\sqrt{\frac{L}{\varepsilon}}\log\frac{L}{\varepsilon}\right)$ probings are required to specify a certified approximation with precision ε (CA_ε) to every convex object, not necessarily polygonal, of perimeter L. The derivation relies on the results proved in the previous sections, which leaves only little to be done in this section.

Theorem 4: Let S be a general convex object with perimeter L. Then, probing according to the proposed probing strategy (with parameter 2ε) guarantees that the height of R_j cannot exceed 2ε and that R_j^c is a certified approximation to S after no more than $j = UB$ probings, where asymptotically

$$UB \approx \sqrt{\frac{\pi}{8}}\sqrt{\frac{L}{\varepsilon}}\log_2\left(\frac{L}{\varepsilon}\right).$$

Proof: Consider the unknown object S. The procedure described in Section II yields a polygon $P(S)$ which circumscribes S and has no more than

$$V \leq \pi\left(\frac{L}{2\pi\varepsilon}+1\right)^{1/2}+1 \tag{43}$$

vertices (Theorem 1). The perimeter of this polygon, L' satisfies the relation $L' < L + 2\pi\varepsilon$ [25]. The results of probing S with (exact) line probes are tangent lines which may be considered also as the results of probing the imaginary polygon $P(S)$ with a positive error line probe ($d = \varepsilon$, see Fig. 12). The results of the probing must be consistent since they are the results of probing a real object. Probing convex polygon according to the proposed strategy yields a set R_j whose height if 2ε or less after n probings, where

$$n \leq V\log_2\left(\frac{128L'}{\varepsilon V}\right). \tag{44}$$

(L' is the perimeter of the polygon, and V is the number of its vertices (Lemma).) Inserting these values for the imaginary circumscribing polygon, it follows that the height of R_j is 2ε or less after no more than

$$n \leq V\log_2\left(\frac{128L'}{\varepsilon V}\right) \leq \left(\sqrt{\frac{\pi L}{2\varepsilon}+1}+1\right) \times \log_2\frac{128(L+2\pi\varepsilon)}{\varepsilon\left(\sqrt{\frac{\pi L}{2\varepsilon}+1}+1\right]} = UB \tag{45}$$

Asymptotically, when $\varepsilon \ll L$.

$$\text{UB} \approx \sqrt{\frac{\pi}{8}}\sqrt{\frac{L}{\varepsilon}}\log_2\left(\frac{L}{\varepsilon}\right). \tag{46}$$

The polygon R_j circumscribes the unknown object S (it includes S, and each of its sides is tangent to it). Hence, By Theorem 2, the distance between any convex object S satisfying the measurements and the central polygon of R_j is guaranteed to be ε or smaller. Thus, R_j^c is a certified approximation of S.

Note that the exact bound is given by (45), whereas (46) gives only an asymptotic approximation. It is not difficult to show that the bound derived in the above theorem is asymptotically tight (up to a multiplicative constant). Consider the case where the unknown object S is a regular polygon with

$$V = c\sqrt{\frac{L}{\varepsilon}}$$

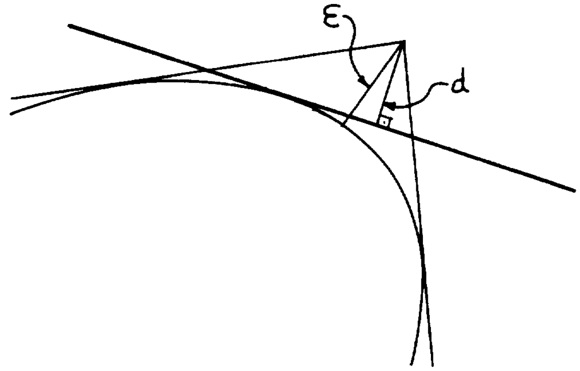

Fig. 12. Probing the true object with a line probe is equivalent to probing the circumscribing polygon with a positive error probe.

vertices (c—constant). The length of each side is

$$L_i = \frac{L}{V} = \frac{1}{c}\sqrt{L\varepsilon}$$

and thus,

$$O\left(\log_2\left(\sqrt{\frac{L}{\varepsilon}}\right)\right)$$

probings are required to lower the height of the corresponding type (ii) vertex to 2ε. Thus,

$$O\left(\sqrt{\frac{L}{\varepsilon}}\log_2\left(\sqrt{\frac{L}{\varepsilon}}\right)\right)$$

probings are required to lower the height of all vertices to the specified value of 2ε and to achieve the desired certified approximation. (Asymptotically, if

$$V = c\sqrt{\frac{L}{\varepsilon}}$$

is sufficiently large then there is a type (ii) vertex in R_j for every side of S, implying that the number of type (ii) vertices is V.)

IX. Discussion

This paper shows how geometric probing can be made into a useful technique for shape estimation from partial sparse measurements. The demand for exact reconstruction is replaced by an easier and practical demand of finding a certified approximation from an adaptive sequence of line probings. A lower bound, LB, on the number of probings required to achieve an approximation with a certain precision, is derived. A probing strategy relying on the basic notion of starting by uniform probing and focusing adaptively on higher uncertainty directions, is proposed. The performance of the proposed strategy is investigated by deriving an upper bound, UB, on the number of probings it requires to guarantee an approximate reconstruction with a certain precision. Both bounds depend on the normalized precision required ε_n defined as $\varepsilon_n = \frac{\varepsilon}{L}$, where ε is the required precision of the approximation and L is the perimeter of the unknown object. It follows that the number of probings, n, required by the optimal strategy satisfies

$$\Omega\left(\sqrt{\frac{1}{\varepsilon_n}}\right) = \mathrm{LB} < n_{\mathrm{optimal}} < \mathrm{UB} = 0\left(\sqrt{\frac{1}{\varepsilon_n}}\log\frac{1}{\varepsilon_n}\right). \tag{47}$$

The lower bound LB is, in fact, a lower bound on the number of sides of a polygon circumscribing a given object S and having a "height" of 2ε or less. By Theorem 2, this is also a bound on the number of probings required to find a certified approximation to S using line probing and complete information about S. This task, which is easier than the one discussed in the paper is the true parallel of the Verification task considered in geometric probing. In the context of Verification, one already has the information about the shape and position of a convex polygon, and uses the probing process to verify the correctness of this information. Here, the information on the object is also given, but only approximate verification is required.

Note that the strategy proposed is similar, in principle, to the hierarchical representations of images [21]. The region quadtree is based on examining a square part of the image, and if it is not uniform according to some measure, it is split to a four equal square cells, and the process is repeated recursively for each of them. This process yields a representation of the image made of square uniform cells whose sizes are adapted to the local uniformity of the image. In the strategy proposed here, the amount of uncertainty is examined for each of the angular intervals whose end points are directions of probings, and if the uncertainty is above a certain level, another probing is done, the interval is halved and the uncertainty is recursively examined for each of its halves.

Real world applications usually involve nonconvex objects, to which our method is not directly applicable. Note, however, that probing nonconvex objects with the (infinite) line probe will reveal their convex hull. Another possible practical extension may come if the object's boundary may be partitioned into a finite number of parts separated by inflection points. Then, if a sensor is capable of specifying a direction and returning the distance of a boundary segment perpendicular to this direction (e.g., ultrasound), then one can use a variation of our strategy for reconstruction.

Many interesting open problems arise. The first obvious one is to close the gap between the lower bound and the proved performance of the proposed strategy. Extending the results to hyperplane probing and to higher dimensions remains an open problem too. Finding certified approximations using different metrics will also lead to completely different problems. It is interesting to note, in this context, that if difference of area is considered as the metric, and if a line probe which reveals the tangency point is used, then an optimal blind approximation method is available [16].

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REFERENCES

[1] P. D. Alevizos, J. D. Boissonnat, and M. Yvinec, "Probing non-convex polygons," in *Proc. IEEE Int. Conf. Robotics Autom.*, Phoenix, AZ, 1989.
[2] H. J. Bernstein, "Determining the shape of a convex n-sided polygon by using $2n + k$ tactile probes," *Info. Processing Lett.*, vol. 22, pp. 255–260, 1986.
[3] A. Bolour and T. M. Cover, "On the number of convex sets on the unit square," Tech. Rep. 2, Dept. of Statistics, Stanford Univ., 1972.
[4] R. Cole and C. K. Yap, "Shape from probing," *J. Algorithms*, vol. 8, pp. 19–38, 1987.
[5] D. Dobkin, H. Edelsbrunner, and C. K. Yap, "Probing convex polytopes," in *Proc. 8th ACM Symp. Theory of Computing*, 1986, pp. 424–432.
[6] H. G. Eggleston, *Convexity*. Cambridge, U.K.: Cambridge University Press, 1977.
[7] H. Edelsbrunner and S. S. Skiena, "Probing convex polygons with x-rays," *SIAM J. Computing*, vol. 17, pp. 870–882, 1988.
[8] L. Fejes Toth, "Approximations by polygons and polyhedra," *Bull. Amer. Math. Soc.*, vol. 54, pp. 431–438, 1948.
[9] J. P. Greschak, "Reconstructing convex sets," Ph.D. dissertation, MIT, Feb. 1985.
[10] P. M. Gruber, "Approximation of convex bodies," in *Convexity and its Applications*, P. M. Gruber and J. M Wills, Eds. Birkhauser, 1983.
[11] W. E. L. Grimson and T. Lozano-Perez, "Model based Recognition and localization from sparse range or tactile data," *Int. J. Robotics Res.*, vol. 3, no. 3, pp. 3–35, 1984.
[12] P. S. Kenderov, "Polygonal approximation of plane convex compacts," *J. Approximation Theory*, vol. 38, pp. 221–239, 1983.
[13] S. Y. R. Li, "Reconstruction of polygons from projections," *Info. Processing Lett.*, vol. 28, pp. 235–240, 1988.
[14] M. Lindenbaum and A. Bruckstein, "Reconstruction of convex polygon from binary perspective projections," *Pattern Recognition*, vol. 23, no. 12, pp. 1343–1350, 1990.
[15] M. Lindenbaum and A. Bruckstein, "Parallel strategies for geometric probing," *J. Algorithms*, vol. 13, pp. 320–349, 1992.
[16] J. S. Lew and D. N. Quarles, Jr., "Optimal inscribed polygons in convex curves," *Am. Math. Month.*, pp. 886–902, 1989.
[17] R. J. McEliece, *The Theory of Information and Coding, A Mathematical Framework for Communication*. Reading, MA: Addison-Wesley, 1977.
[18] D. E. McClure and R. A. Vitale, "Polygonal approximation of plane convex bodies," *J. Math.Anal. Appl.*, vol. 51, pp. 326–358, 1975.
[19] J. L. Prince and A. S. Wilsky, "Reconstructing a convex set from noisy support line measurements," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 12, pp. 377–389, 1990.
[20] T. J. Rivlin, *An Introduction to the Approximation of Functions*. Braisdel, 1969.
[21] H. Samet, *The Design and Analysis of Spatial Data Structures*. Reading, MA: Addison-Wesley, 1989.
[22] S. S. Skiena, "Geometric probing," doctoral dissertation, Department of Computer Science, University of Illinois, Urbana, IL, 1988.
[23] S. S. Skiena, "Problems in geometric probing," *Algorithmica*, pp. 599–605, 1989.
[24] V. M. Tikimorov, "Kolmogorov's work on e-entropy of functional classes and the superposition of functions," *Russian Math. Surveys*, vol. 18, pp. 51–75, 1963.
[25] I. M. Yaglom and V. G. Boltyanskii, *Convex Figures*. New York: Holt, Rinehart and Winston, 1961.
[26] N. V. Zhivkov, "A necessary and sufficient condition for best Hausdorff approximation of plane convex compacta by polygons," *Colloquia Mathematica Societatis Janos Bolai* 48. *Intuitive Geometry*, Siofok, pp. 663–687, 1985.

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