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Antialiasing the Hough Transform[1]

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The discretization of the Hough transform parameter plane is considered. It is shown that the popular accumulator method implies sampling of a nonbandlimited signal. The resultant aliasing accounts for several familiar difficulties in the algorithm. Bandlimiting the parameter plane would allow Nyquist sampling, thus aliasing could be avoided. An effectively alias-free Hough algorithm is presented and analyzed. The uncertainty principle of signal representation induces a compromise between image-space localization and parameter-space sampling density, as well as an upper bound on the performance of the algorithm. These results contribute to the development of a design methodology for hierarchical "coarse to fine" Hough algorithms. © 1991 Academic Press, Inc.

1. INTRODUCTION

In this paper the Hough transform [14, 17, 16] for straight line detection using normal parameters as suggested by Duda and Hart [10] is considered. With the aim of detecting lines through large collinear subsets of a planar set of edge points $P \triangleq \{(x_i, y_i), i = 1, \ldots, N\}$, each point is regarded as a constraint

$$\rho = x_i \cos \theta + y_i \sin \theta \qquad (1)$$

on the normal parameters (ρ, θ) of the straight lines on which the point may be located. Drawn on the (ρ, θ) normal parameters plane, the intersection of a large number of sinusoids corresponds to the normal parameters of a straight line through a large collinear subset of P.

In the standard implementation, (a subset of) the (ρ, θ) parameter plane is divided into $N_\rho \times N_\theta$ rectangular cells, and each cell is represented by an accumulator in an $N_\rho \times N_\theta$ accumulator array. The algorithm is performed in two stages; the first is an incrementation stage in which for each $i \in [1, \ldots, N]$ the accumulators corresponding to cells that the sinusoid (1) intersects are incremented. The second stage is an exhaustive search for maxima in the accumulator array. These represent the normal parameters of straight lines through large collinear subsets of points.

This technique is quite general and has indeed been extended to allow the detection of other parametric shapes, such as circles, ellipses, and parabolas. The use of edge-direction information as a further constraint on the parameters of possible lines, as suggested by O'Gorman and Clowes [34], was instrumental in the generalization of the Hough transform by Ballard [1] to detect arbitrary shapes. Another approach has recently been suggested by Casasent and Krishnapuram [7, 22].

Contributions to the theoretical analysis of the Hough transform have been made by Sklansky [44], Shapiro [40, 41], Shapiro and Iannino [43], Brown [4], Maitre [30], Cohen and Toussaint [8], Van Veen and Groen [50], and others. Deans [9] has shown that the Hough transform can be regarded as a special case of the well known Radon transform.

A distinction is usually made between the study of the performance of the Hough transform with respect to localization accuracy and the research concerning its performance as a detector. Recent Ref. [33] is mostly concerned with localization accuracy issues, while Refs. [12, 13, 23] exemplify current research on detection performance of the Hough transform.

The quantization of the parameter space, inherent in any implementation on a digital computer, is a source for several problems and design trade-offs. Increasing the resolution is usually assumed to lead to better accuracy in the parameters of the detected lines, but to a larger storage requirement and a heavier computational burden. The challenge of this classical trade-off is met by the rapid decline in the cost of memory, by implementing the algorithm on parallel processors, e.g., [42], and in specialized VLSI hardware, e.g, [38, 6], by employing multiresolution "coarse to fine" strategies, e.g., [45, 35, 3, 32, 5, 15, 27, 26, 2], and recently by interpolating the parameter space [33].

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The sucess in pushing the limits of the Hough parameter space resolution has not alleviated several other quantization-related problems. Especially, it is clear that at any finite resolution a high count at an accumulator could result from the combined effect of several insignificant peaks rather than from a single significant one. On the other hand, a true peak could be split between several accumulators and lost. This phenomenon, which induces degradation in the detection performance of the Hough transform, has been studied by Van Veen and Groen [50] as well as by Skingley and Rye [47], Niblack and Petkovic [33], and others. An excellent demonstration of these problems can be obtained by applying the Hough algoritm to detect straight lines in an image which contains a "multiscale curve" [31]. At any fixed resolution false maxima and peak spreading would be apparent.

In their recent paper Niblack and Petkovic [33] considered the Hough transform for straight line detection, and stated the following four open questions:

1. How should the quantization steps $\Delta\rho$ and $\Delta\theta$ of ρ and θ be chosen?
2. Can improved accuracy be obtained by additional preprocessing and/or interpolation of the Hough transform $h(\rho, \theta)$ instead of simply taking the cell with the maximum count?
3. What are the effects of noise in the coordinates (x_i, y_i) on the location of the peak?
4. What is the accuracy achievable using the Hough transform, and how does it compare with that from other techniques, specifically least squares?

The purpose of this paper is to establish a theoretical framework which contributes to the understanding of these issues. Reference [20] is a preliminary version of this paper.

2. ALIASING IN THE HOUGH ALGORITHM

In the Hough algorithm [10] the detection of collinear points is substituted by the detection of sinusoid intersections. The key to the implementation of the algorithm is the two-stage accumulator method for the detection of sinusoid intersections. In the first stage accumulators are incremented—"voted for"—by sinusoids. The second stage is a search for maxima in the accumulator array.

The voting process is intended to produce at the accumulator array a discrete approximation of the continuous-domain Hough transform $h(\rho, \theta)$, defined as follows:

Let $\rho_i \geq 0$ and $0 \leq \theta_i < 2\pi$ denote the polar coordinates of a data point $p_i \in P$. Then every $p_i \in P$ generates a sinusoid $\rho_i^0(\theta)$ in the (ρ, θ) parameter plane:

$$\rho_i^0(\theta) = \rho_i \cos(\theta_i - \theta), \quad \theta \in [0, \pi). \tag{2}$$

An indicator function is associated with each sinusoid:

$$I_i^0(\rho, \theta) = \begin{cases} 1, & \rho = \rho_i^0(\theta) \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Summing up the indicator functions yields the continuous-domain Hough transform:

$$h(\rho, \theta) = \sum_{i=1}^{N} I_i^0(\rho, \theta). \tag{4}$$

The representation of a continuous-domain function by a discrete set of numbers, as is the representation of $h(\rho, \theta)$ by the contents of the accumulator array, is referred to as "digitization" in the signal processing literature, a process which may generally consist of three stages: prefiltering the continuous domain function, sampling it, and quantizing the samples.

Two incrementation rules are commonly used in conjunction with the accumulator method. Either an accumulator is incremented wherever a sinusoid traverses the cell in the normal parameters plane to which that accumulator corresponds, or alternatively a rectangular grid is imposed on the (ρ, θ) plane, each accumulator corresponding to a grid point; an accumulator is incremented wherever the respective grid point is nearest to an intersection of a sinusoid with a grid line parallel to the ρ axis. In signal processing terms both incrementation rules are equivalent to a certain space-variant transformation followed by sampling on a rectangular grid.

$h(\rho, \theta)$ is, however, a discontinuous—hence nonbandlimited—function. This is hardly changed by the spatial-dependent transformation inherent in the voting process. It is well known that due to aliasing effects a nonbandlimited signal cannot be properly represented by a discrete set of samples. This indicates a basic inadequacy in the implementation of the Hough algorithm.

Various aspects of this problem are studied in the rest of this paper, and a remedy is developed and analyzed. A key to any solution should involve, however, the replacement of $h(\rho, \theta)$ by an essentially bandlimited function whose Nyquist rate is finite, such that sufficient parameter-plane sampling could be carried out. Interpolation by an appropriate low-pass filter would then allow to search for maxima at any desired resolution.

"Blurring" the accumulator array in the process of voting has indeed been suggested, e.g., by Niblack and Petkovic [33]. Other authors suggested to smooth the accumulator array following the incrementation stage; this is less effective. Niblack and Petkovic [33] have also insightfully employed interpolation in the parameter plane to increase the effective resolution. However, to the best of our knowledge, no previous author has taken a frequency-domain approach to guide the discretization of the Hough transform parameter space.

3. STRAIGHT LINE DETECTION AS AN OPTIMIZATION PROBLEM

The Hough transform is aimed at detecting collinear subsets within a planar set of points, which usually result from the application of edge detection and thresholding to an image of a scene. Some points result from straight edges in the underlying scene; other points are considered as "noise."

Let us consider a simplified situation, where all points are the experimental outcome of a single straight edge. Due to various random effects these points will not be truly collinear. Furthermore, the points generally appear as small "blobs." Thus, the design of an algorithm to extract the parameters of the straight line passing through these points requires a measure of fitness such that lines could be compared and an "optimal" line could be selected.

If the deviation of every data point from its "true" position could be considered a random variable with a known distribution, then a meaningful measure of fitness in the statistical sense could be devised; otherwise its selection is quite arbitrary. A common choise is to take the line that minimizes the sum of squared distances to the data point, the distances being measured either in parallel to one of the axes or normal to the line. See [11].

The method of least squares is inadequate when data points due to noise and to other edges are present. This problem has recently been treated by several authors, e.g., Weiss [51], Kamgar-Parsi and Kamgar-Parsi [24, 25], Otsu [36], Thrift and Dunn [48], and Kiryati and Bruckstein [19, 21]. The approach of [48, 19] will be followed here because of its close relation with the Hough algorithm.

Each data point contributes to the "weight" of every candidate line according to an "influence function" which relates the contribution to the normal distance between the line and the point. The "best" line is the line that has maximum weight. The influence function could be specified as desired, but a meaningful choice is a positive, monotonically decreasing function of the distance. (For mathematical convenience the influence function is defined to be symmetric, and is usually a decreasing function of the *absolute value* of its argument.) By choosing such a function, whose *localization* can be represented by an "effective radius" proportional, perhaps, to its second moment, the designer indeed implies that lines passing within the effective radius are considered as related to the point, while other lines are not. A scalar, circularly symmetric "influence field" can be visualized around each data point: the contribution of the point to the weight of a line is the value of the field at the foot of the normal from the line to the point.

The special relation between this line fitting approach and the Hough algorithm stems from the following property:

Let the sinusoid (2) represent the locus of the normal parameters of all possible straight lines through a data point $p_i = (x_i, y_i)$. The locus of all straight lines tangent to a circle of radius r around p_i, i.e., the locus of all lines whose distance to p_i is r, is a pair of dc-biased sinusoids in the (ρ, θ) plane:

$$\rho_i^+(\theta) = r + \rho_i^0(\theta) = r + \rho_i \cos(\theta_i - \theta), \quad \theta \in [0, \pi) \tag{5a}$$

$$\rho_i^-(\theta) = -r + \rho_i^0(\theta) = -r + \rho_i \cos(\theta_i - \theta), \quad \theta \in [0, \pi). \tag{5b}$$

An extended Hough technique to determine lines of high weight which is based on the above-mentioned property can now be outlined. Let $c(r)$ denote the (symmetric) influence function, and let an extended indicator function $I_i(\rho, \theta)$ represent at every pair (ρ, θ) the weight contributed by the point p_i to the line. From the property,

$$I_i(\rho, \theta) = c(\rho - \rho_i^0(\theta)). \tag{6}$$

Note that for every fixed θ, $I_i(\rho, \theta)$ is a convolution of the influence function $c(\rho)$ with an impulse $\delta(\rho - \rho_i \cos(\theta_i - \theta))$. The total weight accumulated by a line whose normal parameters are (ρ, θ) is

$$z(\rho, \theta) = \sum_{i=1}^{N} I_i(\rho, \theta). \tag{7}$$

Note that the extended Hough transform $z(\rho, \theta)$ can be visualized as the Radon transform of a modified input function in which each data point is replaced by a circularly symmetric density distribution which is the inverse Abel transform of the influence function.

Significant lines can be detected by employing the accumulator method to evaluate samples of $z(\rho, \theta)$ and search for peaks. Clearly, more accumulators need to be incremented than in the conventional Hough algorithm; the exact number depends on the support of the influence function. The key to efficient implementation of the computationally critical accumulation stage is the systematic incrementation law (6). It is easily observed that for every discrete value of θ, a vector which is the discretization of $c(\rho - \rho_i^0(\theta))$ must be added to the respective column of the accumulator array. This operation can be carried out very efficiently by many modern computers optimized for image processing tasks. Furthermore, moving to the next value of θ requires just a shift of the contents of the vector according to $\rho_i^0(\theta)$, e.g., by relative indexing. In a purely serial implementation the number of operations in the accumulation process would linearly depend on the support of the influence function.

Using the influence function

$$c^H(r) = \begin{cases} 1, & \rho = 0 \\ 0, & r \neq 0 \end{cases} \tag{8}$$

reduces this algorithm to the conventional Hough algorithm. Note that by using the influence function

$$c^S(r) = \begin{cases} 1, & |r| \leq d \\ 0, & |r| > d \end{cases} \tag{9}$$

Shapiro's algorithm [41] for detecting straight lines in the presence of isotropic quantization errors (limited by d) is obtained.

4. BAND REGION OF THE EXTENDED HOUGH TRANSFORM

It is obvious that the extended Hough transform $z(\rho, \theta)$ is not bandlimited if general influence functions are allowed. Hence, an implementation of the algorithm based on the accumulator method implies aliasing. For example, implementing Shapiro's algorithm [41], that specifies the discontinuous influence function (9), calls for sampling of a function $z(\rho, \theta)$ consisting of sinusoidal bands whose vertical profiles are a rectangular pulse. Thus $z(\rho, \theta)$ is discontinuous, nonbandlimited, and its sampling clearly results with aliasing. It can nevertheless be shown that the situation is somewhat improved with respect to the conventional Hough transform, because the magnitude of the Fourier transform of (9) is upper-bounded by a decreasing function of the frequency, while the magnitude of the Fourier transform of (8) is constant.

If, however, it is possible to specify a special influence function that leads to an effectively bandlimited extended Hough transform while retaining adequate localization in a well defined sense, then an essentially *alias-free Hough transform* can be devised.

In this section the bandwidth of the extended Hough transform is computed under the assumption that a bandlimited influence function is used. The calculation and its results are analogous to the Radon transform bandwidth computation performed by Rattey and Lindgren [39, 29].

The extended Hough transform $z(\rho, \theta)$ of the planar set of points P can be expressed as

$$z(\rho, \theta) = \sum_{i=1}^{N} c(\rho - \rho_i^0(\theta)) = \sum_{i=1}^{N} c(\rho - \rho_i \cos(\theta - \theta_i)). \tag{10}$$

The linearity of the Fourier transform implies that $Z(w_\rho, w_\theta)$, the 2-D Fourier transform of $z(\rho, \theta)$, can be expressed as

$$Z(w_\rho, w_\theta) = \sum_{i=1}^{N} Z_i(w_\rho, w_\theta), \tag{11}$$

where $Z_i(w_\rho, w_\theta)$ denotes the 2-D Fourier transform of the 2-D composite function $c(\rho - \rho_i \cos(\theta - \theta_i))$:

$$Z_i(w_\rho, w_\theta) = \iint_{-\infty}^{\infty} c(\rho - \rho_i \cos(\theta - \theta_i)) e^{-j\rho w_\rho} e^{-j\theta w_\theta} \, d\rho \, d\theta. \tag{12}$$

Let $C(w)$ denote the 1-D Fourier transform of the symmetric influence function $c(r)$. Integrating with respect to ρ, it is noted that

$$Z_i(w_\rho, w_\theta) = C(w_\rho) \mathcal{F}_\theta\{e^{-j\rho_i w_\rho \cos(\theta - \theta_i)}\} \tag{13}$$

$\exp(-j\rho_i w_\rho \cos(\theta - \theta_i))$ is periodic, thus its Fourier transform is discrete:

$$Z_i(w_\rho, w_\theta) = C(w_\rho) \cdot 2\pi \sum_{k=-\infty}^{\infty} a_k \, \delta(w_\theta - k), \tag{14}$$

where $\delta(\cdot)$ is the impulse function, and

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jk\theta} e^{-j\rho_i w_\rho \cos(\theta - \theta_i)} \, d\theta. \tag{15}$$

A straightforward calculation yields

$$a_k = e^{-jk(\pi/2 + \theta_i)} J_k(\rho_i w_\rho), \tag{16}$$

where $J_k(\rho_i w_\rho)$ denotes the order-k Bessel function of the first kind. Substituting (16) into (14) results in

$$Z_i(w_\rho, w_\theta) = C(w_\rho) \cdot 2\pi \sum_{k=-\infty}^{\infty} e^{-jk(\pi/2 + \theta_i)} \cdot J_k(\rho_i w_\rho) \, \delta(w_\theta - k). \tag{17}$$

Assume that the retina is finite, i.e., $\rho_i < \rho_M, \forall i$, and a bandlimited influence function such that $C(w_\rho)$ is nearly

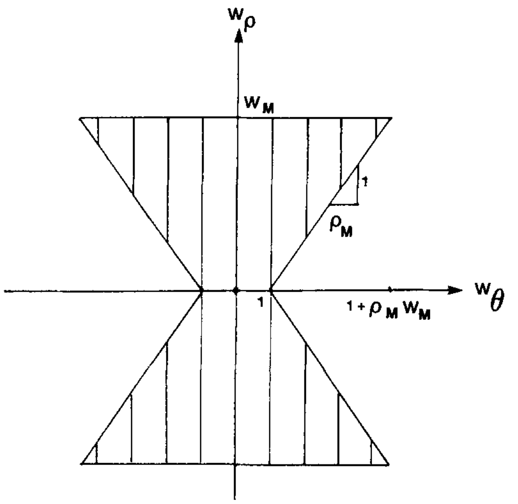

Fig. 1. The "$\rho_M w_M$ bow tie" effective region of support of $Z(w_\rho, w_\theta)$.

zero for $w_\rho > w_M$. A comparison with [39] reveals that $Z(w_\rho, w_\theta)$ is mathematically equivalent to the 2-D Fourier transform of the Randon transform of a function whose effective region of support is a circle of radius ρ_M, and which is also effectively bandlimited to a circle of radius w_M.

$J_k(\beta)$, the Bessel function of the first kind of order k and argument β, decays monotonically for $k/\beta > 1$; furthermore $|J_k(\beta)| \ll 1$ if $|k/\beta| \gg 1$. To calculate the effective support of $Z(w_\rho, w_\theta)$ Rattey and Lindgren applied a standard approximation from FM-communication theory that $J_k(\beta)$ is effectively zero for

$$|k| > |\beta| + 1. \tag{18}$$

Under this approximation they have found the region of support of $Z(w_\rho, w_\theta)$ to be effectively confined to a "$\rho_M w_M$ bow tie," as shown in Fig. 1.

5. SAMPLING REQUIREMENTS FOR THE EXTENDED HOUGH TRANSFORM

After showing in the previous section that a bandlimited influence function leads to a bandlimited extended Hough transform whose band region is a "$\rho_M w_M$ bow tie," it is now necessary to determine the sampling requirements that would guarantee an alias-free Hough transform.

Assuming that the influence function is effectively zero for $|r| > r_M$, it is noted that the support of the extended Hough transform in the ρ direction is limited to

$$|\rho| \leq \rho_M + r_M. \tag{19}$$

Since the extended Hough transform satisfies

$$z((-1)^k \rho, \theta + k\pi) = z(\rho, \theta), \tag{20}$$

then, provided that the number of samples in the θ direction is even, it needs to be sampled only for $\theta \in [0, \pi)$.

Rattey and Lindgren [39] have shown that optimal sampling for signals whose band region is the "bow tie" shown in Fig. 1 is on an hexagonal grid. They have determined that the intersample distance in the ρ direction must satisfy

$$\Delta\rho < \pi / w_M \tag{21}$$

and with hexagonal sampling

$$\Delta\theta \leq \pi / (\lfloor \rho_M w_M \rfloor + 3), \tag{22}$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than x. The resultant minimum required number of samples is

$$L = (\rho_M + r_M) w_M (\lfloor \rho_M w_M \rfloor + 3)/\pi. \tag{23}$$

Normalizing (23) by assuming without loss of generality $\rho_M \triangleq 1$ gives

$$L = (1 + r_M) w_M (\lfloor w_M \rfloor + 3)/\pi. \tag{24}$$

If a reasonably localized influence function satisfying $r_M \ll \rho_M$ is assumed, (23) degenerates to the sampling requirement for the "space and bandlimited" Radon transform [39]. $r_M \ll \rho_M \triangleq 1$ also implies $w_M \gg 1$, allowing the approximation

$$L \cong w_M^2 / \pi. \tag{25}$$

Sampling the extended Hough transform on a rectangular grid instead of on the optimal hexagonal grid results in a doubled sampling requirement.

6. THE COMPROMISE BETWEEN IMAGE-PLANE LOCALIZATION AND SAMPLING REQUIREMENTS

The subject of this section is the design of (symmetric) influence functions that meet two objectives simultaneously. First, they must be well localized in the spatial domain within an interval $(-r_M, r_M)$ to allow adequate localization in the image plane. Second, their Fourier transform must be adequately localized within an interval $(-w_M, w_M)$ to decrease the number of samples required to obtain negligible aliasing.

The uncertainty principle of signal representation [37] dictates, however, that the influence function $c(r)$ and its

Fourier transform $C(w)$ cannot both be of "short duration." Depending on the meaning of the word "duration" the uncertainty principle can take several mathematical forms. Since any implementation of the extended Hough transform implies an influence function of finite duration, it seems meaningful to focus on designing influence functions that are truly space limited in $(-r_M, r_M)$ while having the smallest possible "effective bandwidth" in a certain sense. (As mentioned earlier, the number of operations in a purely serial implementation of the accumulation process linearly depends on r_M).

Letting E denote the energy of the influence function

$$E = \int_{-r_M}^{r_M} c^2(r)\, dr = \frac{1}{2\pi} \int_{-\infty}^{\infty} C^2(w)\, dw \tag{26}$$

and D^2 denote the second-order energy moment of $C(w)$

$$D^2 = \frac{1}{2\pi E} \int_{-\infty}^{\infty} w^2 |C(w)|^2\, dw, \tag{27}$$

it is known [37] that

$$r_M \cdot D \geq \pi/2 \tag{28}$$

and that equality holds only for the influence function

$$c(r) = \begin{cases} k \cos(\pi r/2r_M), & |r| \leq r_M \\ 0, & |r| > r_M, \end{cases} \tag{29}$$

where k is a constant. With $k > 0$ this influence function is a positive, symmetric, and monotonically decreasing function of $|r|$ within its interval of support.

For sampling purposes, an effective bandwidth w_M can be defined as proportional to D. A large enough proportionality constant would ensure negligible aliasing. A reasonable definition is $w_M = 3D$, corresponding in the optimal function (29) to the width of the main lobe of its Fourier transform. See Fig. 2.

Alternatively, it is possible to design influence functions that minimize w_M, now defined as the bandwidth into which a certain fraction β of the energy is confined, i.e.,

$$\beta = \frac{1}{2\pi E} \int_{-w_M}^{w_M} |C(w)|^2\, dw. \tag{30}$$

The relevant version of the uncertainty principle is related to prolate spheroidal functions [46, 28, 37] which are the solutions of the eigenvalue–eigenfunction problem

$$\int_{-r_M}^{r_M} \varphi(x) \frac{\sin w_M(r-x)}{\pi(r-x)}\, dx = \lambda \varphi(r). \tag{31}$$

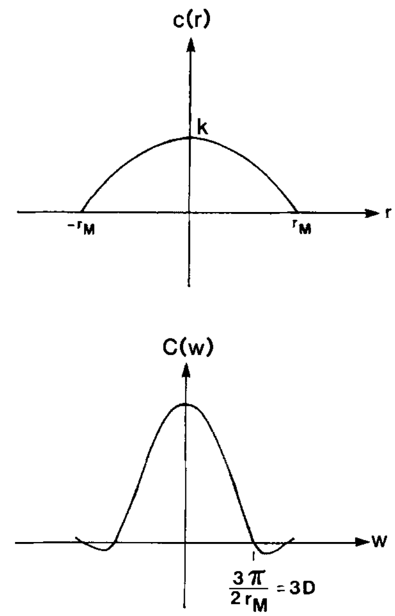

Fig. 2. The influence function $c(r)$ that achieves the uncertainty bound (28) and its Fourier transform.

In particular,

$$r_M w_M \geq f(\beta), \tag{32}$$

where $f(\beta)$ is a positive, monotonically increasing function, and equality holds only for the influence function

$$c(r) = \begin{cases} k\varphi_0(r), & |r| \leq r_M \\ 0, & |r| > r_M, \end{cases} \tag{33}$$

where k is a constant and $\varphi_0(r)$ is the eigenfunction of (31) which corresponds to its largest eigenvalue. The uncertainty relation (32) also demonstrates the relation between the sampling requirement, which depends on w_M, and the selection of the threshold under which aliasing is assumed negligible, as implied by β. Note that the Kaiser window [18] is an excellent approximation of the prolate spheroidal function.

7. CONCLUSIONS

In this section conclusions are drawn from the results of this research toward a better understanding of the four open problems which have been presented by Niblack and Petkovic [33] and mentioned in the Introduction to this work, and toward strengthening the theoretical foundation for multiresolution focusing Hough transform methods.

One question posed in [33] is whether or not improved accuracy can be obtained by preprocessing and/or interpolation of the Hough transform instead of by simply taking the cell with the maximum count. The answer to this question is affirmative. In particular, the preprocessing advocated here is to alleviate the aliasing problem inherent in the accumulator method by employing the extended Hough transform [48, 19] in conjunction with carefully designed influence functions that are effectively space- and bandlimited, providing the best compromise between image-domain accuracy and transform-domain bandwidth. Low-pass filtering, in correspondence with the size and the shape of the transform's bandregion, is the optimal postsampling interpolation scheme, but even local, computationally economical interpolators would yield better results than simply taking the cell with the maximum count, equivalent to crude zero-order hold interpolation.

Another question is how should the quantization steps $\Delta\rho$ and $\Delta\theta$ be chosen. The suggested reply is that to allow effectively alias-free representation of the transform by its samples the sampling intervals $\Delta\rho$ and $\Delta\theta$ should be chosen to satisfy the Nyquist condition. The optimal sampling grid is hexagonal rather than rectangular; the required sampling intervals and the total required number of samples are given by inequalities (21), (22), and (25) in terms of the effective space and bandwidth of the influence function, which should be designed to correspond with the desired image-domain accuracy.

To complete the answer to this question several remarks are due. First, the accuracy of the input data points provides a lower bound on the useful value of r_M, the effective radius of the influence function's spatial support, and through the uncertainty principle an upper bound on its useful effective bandwidth. If the total number of sampling points is severely limited, image-domain accuracy must be sacrificed by specifying a narrow-band influence function whose effective radius in the image domain is large with respect to the accuracy of input data. In this case the performance of the Hough transform is limited by the resources rather than by the quality of the data. If, however, it would be possible to increase the number of sampling points, the effective radius of the influence could be made to correspond with the accuracy of the data, allowing the performance of the transform to reach the limit posed by the accuracy of the data. Increasing the sampling density even further (without unnecessarily increasing the bandwidth of the influence function) would have the effect of producing guard-bands between the replicas of the transform's bandregion, thus allowing the specifications of the interpolating low-pass filter to be relaxed, and making computationally economical "local" interpolation feasible.

A further question presented in [33] concerns the effects of noise in the coordinates (x_i, y_i) of the input data points on the location of the peak. The sensitivity of the Hough transform in its usual definition to errors ("noise") in the coordinates of the input data points is well known. In particular, it has the effect of smearing the peak in the transform domain between several accumulator cells, sometimes to such a degree that the peak is lost in the background. The extended Hough transform with a suitable influence function whose spatial extent reflects the accuracy of the data has indeed evolved [41, 48, 19] as a remedy to this problem as much as can be allowed by the quality of the data. If properly applied, the resultant peak in the transform domain is indeed smooth, but sufficient sampling and proper interpolation allow accurate peak detection and parameter extraction.

The last question concerns the accuracy achievable using the Hough transform, and its comparison with that from other techniques, specifically least squares. In a previous work [19] it has been shown that by specifying various influence functions the extended Hough transform can be made to simulate many line fitting techniques, the least squares (in the normal direction) in particular. Hence, leaving computational considerations aside, the extended Hough transform can be tuned to be equivalent to other methods. Furthermore, by specifying appropriate influence functions, the extended Hough transform easily overcomes the two main obstacles that render many other techniques useless—the presence of input data points due to more than one line, and due to background noise processes.

The accuracy of the extended Hough transform is limited either by the inherent errors in the input data, or by insufficient resources which dictate the employment of influence functions whose spatial extent is large with respect to the accuracy of the data. In the latter case the accuracy achievable using the extended Hough transform is governed by the uncertainty principle of signal representation (Eqs. (28) and (32)) which relates (through the sampling requirement (25)) the number of available accumulators to the spatial extent of the best influence functions.

The total number of available accumulators is severely limited in low-resolution stages of multiresolution Hough transform algorithms. Indeed, significant undersampling-related difficulties have been reported [16] in the application of the "adaptive Hough transform" [15] to complex images. The results presented in this paper provide a clear limit on the achievable resolution in the image domain as a function of the number of available accumulators. A multiresolution extended Hough transform that actually achieves this limit can be devised such that at any stage the spatial extent of the influence function decreases and the sampling requirement increases with respect to the previous stage, approaching the input data quality limit.

Useful multiresolution Hough algorithms are based on focusing—i.e., on combining all resources to perform high-resolution accumulation and search just in small areas of interest in the transform domain which were found in a preceding low-resolution stage of the algorithm. An inherent difficulty arises since in postsampling interpolation of the transform by low-pass filtering the contributions of sampling points throughout the parameter plane are required for producing any interpolated value. This problem can be alleviated by oversampling the transform with respect to the Nyquist rate, thus providing guardbands between replicas of the transform's bandregion and allowing relaxed filtering requirements, which correspond to local interpolation within the focused-on area.

8. DISCUSSION

The accumulator method which is essential in most implementations of the popular Hough algorithm has been regarded here as a peculiar sampling scheme which is applied to the nonbandlimited Hough transform function. To minimize the resultant aliasing errors, the Hough transform should be bandlimited prior to its sampling. Point-sampling has the advantage that the bandlimited Hough transform needs to be evaluated just at the sampling points. Bandlimiting the Hough transform by computational low-pass filtering is impractical, since the filtered value at any sampling point depends on the value of the (continuous-domain) input everywhere. Furthermore, it is not obvious how the performance of the Hough transform in the image-domain is changed by general two-dimensional filtering in the transform domain.

To overcome these problems, the application of an extended Hough transform [48, 19] is suggested. It has been shown in this paper that the extended Hough transform can be tuned by an appropriate choice of an "influence function" to be essentially bandlimited while retaining predictable performance in the image-domain. In particular, the trade-off between the total number of available accumulators L which sets the maximum bandwidth and image-domain performance has been pointed out.

To illustrate, consider the arrangement of data points shown in Fig. 3a. Given the number of accumulators L, one wishes to know whether or not the Hough algorithm can resolve the individual line segments. The theory presented in this paper can provide a straightforward answer to such questions.

For values of L which are not extremely small, satisfying the Nyquist sampling requirement (25) implies that the Fourier transform of the chosen influence function must effectively vanish for

$$|w| > w_M \cong \sqrt{\pi \cdot L}. \tag{34}$$

A reasonable type of influence function is (29). Applying the uncertainty principle (28) and the definition $w_M = 3D$ yields

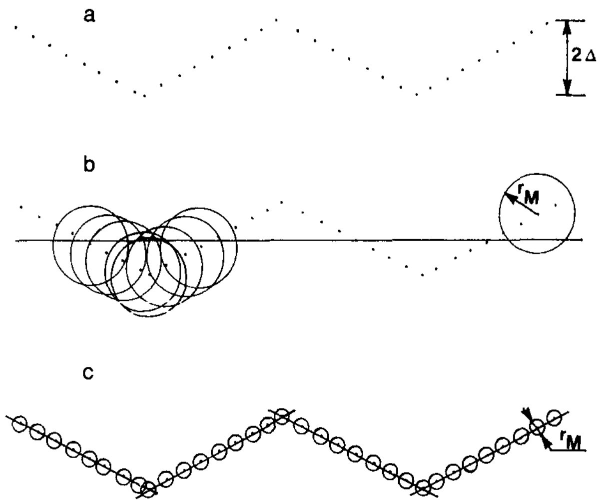

Fig. 3. (a) An arrangement of input points lying on individual line segments. (b) r_M is relatively large, thus the points are interpreted as lying on a single straight line. (c) r_M is sufficiently small, enabling the detection of the individual segments.

$$r_M = \pi/2D = 3\pi/2w_M. \tag{35}$$

Substituting (34) in (35) gives

$$r_M = \frac{3}{2}\sqrt{\frac{\pi}{L}} \cong 2.66/\sqrt{L}, \tag{36}$$

where r_M, which is the radius of support of the (one-dimensional) influence function, can be interpreted in the image domain as the "radius of influence" of a data point: a line is considered by the extended Hough transform to pass "through" a point, if it intersects a circle of radius r_M around the point. Thus, (36) quantifies the trade-off between image-domain localization and the number of accumulators. Referring to Fig. 3a, it is now clear that if $\Delta \approx r_M$ and certainly if $\Delta < r_M$ the extended Hough transform cannot resolve the individual segments, and the data points would be interpreted as lying on a single straight line as shown in Fig. 3b. If however L is large enough such that $\Delta \gg r_M$ the individual segments can be detected: see Fig. 3c.

Possibilities for future research include the extension of the results presented here to versions of the Hough transform other than that of Duda and Hart [10]. These include algorithms that employ gradient direction and magnitude, algorithms for detecting parametric shapes other than straight lines, e.g., circles and parabolas, and algorithms for detecting arbitrary shapes.

The influence functions which were considered in this paper are truly space-limited functions which are essentially—but not absolutely—bandlimited. Thus, a small aliasing error in the sampling of the Hough transform is not completely avoided. By specifying influence functions that are effectively—but not truly—space-limited a truncation error [49] would be introduced, but the aliasing error could be reduced. Balancing the trade-off to obtain an influence function that minimizes the total combined aliasing and truncation error can lead to further enhancement in the performance of the algorithm. This possible improvement must also be traded against the increased number of operations (in a serial implementation of the accumulation process) implied by the increased spatial support of the influence function.

In this paper a parameter plane sampling requirement is presented which is based on the sampling theorem and is related to the effective bandwidth of the extended Hough transform. A subtle observation is that (given a certain influence function) all possible realizations of the extended Hough transform are a priori known to belong to a rather limited subclass within the general class of functions of comparable bandwidth. This motivates work toward finding influence functions that could allow one to determine the extended Hough transform by a smaller number of samples than is required by the sampling theorem.

High-accuracy line detection requires reconstruction of the extended Hough transform from its samples by interpolation, and peak detection in the continuous domain. "Ideal low-pass filter" interpolation, which is required if sampling is carried out near the Nyquist rate, and the associated peak detection algorithm are computationally expensive. Sampling at a higher rate allows the specifications of the interpolation scheme to be relaxed and computationally economical interpolation and peak detection algorithms to be employed. It would be very nice if an interpolation scheme—in conjunction with a sampling requirement and a peak detection algorithm—could be developed that would allow one to locate peaks of the continuous extended Hough transform function directly from its samples, without having to actually carry out the interpolation and to locate the peak by search. A peak detection algorithm in which the value of the peak is a weighted average of nearby samples and its location is their "weighted" center of mass is especially desirable. One of the interpolation methods of [33] can be regarded as a step in this direction.

In this paper the relation between the number of accumulators and the performance of the Hough transform in the image domain has been clarified. Referring to multiresolution "focusing" Hough algorithms, it is clear that a peak in a low-resolution stage can result from the combined effects of several insignificant peaks, while a true peak can be lost in the background. This poses a severe difficulty in the design of focusing Hough algorithms. The development of a focusing strategy that is optimal with respect to a "reasonable" distribution of lines in images is of great theoretical and practical importance and will be the subject of future work.

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