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Uniform multi-agent deployment on a ring☆

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ABSTRACT

We consider two variants of the task of spreading a swarm of agents uniformly on a ring graph. Ant-like *oblivious* agents having limited capabilities are considered. The agents are assumed to have little memory, they all execute the same algorithm and no direct communication is allowed between them. Furthermore, the agents do not possess any global information. In particular, the size of the ring (n) and the number of agents in the swarm (k) are unknown to them. The agents are assumed to operate on an unweighted ring graph. Every agent can measure the distance to his two neighbors on the ring, up to a limited range of V edges.

The first task considered, is dynamical (i.e. in motion) uniform deployment on the ring. We show that if either the ring is unoriented, or the visibility range is less than $\lfloor n/k \rfloor$, this is an impossible mission for the agents. Then, for an oriented ring and $V \geq \lceil n/k \rceil$, we propose an algorithm which achieves the deployment task in optimal time. The second task discussed, called quiescent spread, requires the agents to spread uniformly over the ring and stop moving. We prove that under our model, in which every agent can measure the distance only to his two neighbors, this task is impossible. Subsequently, we propose an algorithm which achieves quiescent but only almost uniform spread.

The algorithms we present are scalable and robust. In case the environment (the size of the ring) or the number of agents changes during the run, the swarm adapts and re-deploys without requiring any outside interference.

1. Introduction

In this paper we consider multi-agent formation problems on a ring graph. The ring consists of n nodes and the number of agents will be denoted by k. Simple (ant-like) agents having limited capabilities are considered. The agents are assumed to be memoryless, all of them execute the same algorithm and no communication is allowed between them. The agents are *uniform* in the sense that they have no identifiers hence cannot be distinguished and they all follow the same algorithm. We also assume that the agents do not possess any global information i.e. the size of the ring (n) and the size of their swarm (k) are unknown to the agents. The agents operate on an unweighted ring graph. Every agent can measure the distance to his two neighbors on the ring up to a limited "visibility" range of V edges: if two adjacent agents are farther than V edges apart, they cannot measure the distance between them. However, in this case, they know that the distance is larger than V.

We consider *oblivious* algorithms. An algorithm is *oblivious* if the action performed by an agent is dependent solely on the system state at the time the action is taken. In particular, each agent's current action is not dependent on past system states.

The algorithms we propose are scalable and robust, in the sense that if the environment (the size of the ring) or the number of agents change during the run, the swarm adapts and re-forms without requiring any outside interference.

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A question that naturally arises is the joint timing or synchronicity of the agents' operations in the environment. We use two synchronicity models by Suzuki et al. [16]. In the *semi-synchronous* model (SSM) the agents operate in cycles. In every cycle only a subset of the agents are active. It is however guaranteed that in an infinite run, every agent will become active infinitely many times. In a time cycle, every active agent senses the environment and takes an action, either traversing an edge or staying in place. In this model, traversing an edge is assumed to be instantaneous. The *synchronous* model (SM) is similar to SSM but in every cycle all agents are active. It is important to distinguish SSM from CORDA [26] (Wait-Observe-Compute-Move). In SSM (and SM) the observe-compute-move cycle is atomic while in CORDA a finite (but unbounded) time may pass between sensing (the "observe" phase) and acting (the "move" phase). Hence, in CORDA, the agents may take an action (move) according to outdated information.

All the impossibility results presented in this paper are proved under SM. Since SM is a particular instance of SSM, the impossibility proofs hold for SSM as well. Furthermore, SM is also a particular instance of CORDA so our impossibility results do hold for CORDA. All the algorithms we shall present work under both SM and SSM. Time bounds for SSM are presented in terms of *rounds* as defined below. Under SM, every time cycle is a *round*. So if algorithm A converges within t *rounds* under SSM then it converges within t time cycles under SM.

Definition 1 (*Round*)**.** A *round* is a time period in which every agent was active at least once.

The tasks considered in this paper are two variants of the task of spreading a group of agents uniformly on a ring graph. Assuming the graph contains n vertices and the swarm is composed of $k \leq n$ agents, the most uniform spread possible is when the distance between any two adjacent agents is either $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$.

Since the agents are indistinguishable and memoryless (i.e. are *oblivious*), a complete system description is given by the agents' location. Formally, a system configuration is a vector of length k specifying the agents' location.

Definition 2 (*Balanced Configuration*)**.** The agents form a balanced configuration if the distance between any two adjacent agents is either $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$.

The first task we consider is forming and maintaining balanced configurations i.e. the agents are required to form a balanced configuration and to stay in a balanced configuration. The agents may move and change configurations as long as they maintain balanced configurations. We denote the task of forming and maintaining a balanced configuration by *uniform spread*.

We first show that in the case of an unoriented ring i.e. the agents do not a priori agree on an orientation of the ring, no deterministic *oblivious* algorithm can consistently achieve a *uniform spread*. Then, considering an oriented ring, we show that if the agent's visibility range (V) is strictly less than $\lfloor n/k \rfloor$, no *oblivious* algorithm can achieve a *uniform spread*. Algorithm 1 presented in Section 4.2 is then proved to achieve a *uniform spread* for $V \geq \lfloor n/k \rfloor$, therefore the analysis covers all possible values of V for the oriented ring. In case $V \geq \lceil n/k \rceil$, a swarm of agents running Algorithm 1 will achieve a *uniform spread* within $O(n)$ time cycles under SM and $O(n)$ *rounds* under SSM. In case all agents start from the same vertex, a *uniform spread* cannot be achieved faster than $\Omega(n)$, hence the algorithm convergence time is optimal. The term oriented or unoriented ring is used for convenience. However, sensing (or agreeing upon) the orientation of the ring can be a capability of the agents and not a characteristic of the ring.

The second task we consider is quiescent *uniform spread* i.e. the agents are required to form a balanced configuration and stop moving. Stopping might be of importance since moving consumes energy. We show that under our *oblivious* model in which every agent can sense only the distance to his two neighbors, forming a quiescent balanced configuration is impossible. Hence we introduce Algorithm 2 under which the agents form quiescent, but only "semi-balanced" configurations.

Definition 3 (*Semi-Balanced Configuration*)**.** The agents form a semi-balanced configuration if the distance between any two adjacent agents is at least $\frac{n}{k} - \frac{k}{2}$ and at most $\frac{n}{k} + \frac{k}{2}$.

In case $n/k \gg k$, a semi-balanced configuration is (almost) balanced. Algorithm 2 achieves quiescent semi-balanced configurations within $O(n)$ *rounds* (or time cycles). In case all agents start from the same vertex, a semi-balanced configuration cannot be formed faster than $\Omega(n)$. Hence the convergence time of Algorithm 2 is optimal. We note that Algorithm 2 requires a slightly larger visibility range, that is $V \geq \frac{n}{k} + \frac{k}{2}$.

Consider several agents on the same vertex. Since the agents are indistinguishable, the symmetry between them cannot be broken and they cannot be separated. We bypass this symmetry breaking problem by assuming distinct initial locations i.e. it is assumed that when the algorithm is initialized there are no two agents occupying the same vertex. Note that under the algorithms we present, if initially the agents were all at distinct vertices then there will never be two agents on the same vertex.

2. Related work

If instead of a ring graph we consider the environment a continuous circle on the plane, we obtain a continuous-space variant of our problem. The uniform circle formation problem is an instance of the general pattern formation problem, in which the agents are required to form a specific geometric constellation on the plane [16]. In one instance of the uniform circle formation problem, the agents are required to uniformly spread over some circle on the plane, and the circle is not predetermined. The circle formation task may be divided into two sub-tasks: in the first sub-task the agents are required to form a non-uniform circle and, in the second, to spread uniformly over that circle.

Sugihara and Suzuki [27] suggested a heuristic algorithm for the limited visibility case. Later, Suzuki and Yamashita [28] proposed a non-*oblivious* algorithm in which the agents had the capability to remember past system states. Defago and Souissi [6] suggested the agreed-upon circle to be the smallest circle enclosing all agents. Their algorithm requires full visibility, i.e. every agent sees all other agents and can perform global calculations. In their solution the agents converge toward a uniform cycle but do not actually reach it. Later, Chatzigiannakis et al. [4] simplified the algorithm proposed by Defago by modifying the model. Katreniak [18] solved the problem of forming a *bi-angular* circle. Dieudonne et al. [9] presented an algorithm which forms a uniform cycle starting from a *bi-angular* circle for any number of agents but 4. Recently, Flocchini et al. [13] considered ϵ-*approximate* solutions. They proved that an exact solution is impossible if the ring is not oriented. A different but related problem is spreading the agents uniformly over a line segment, see [29,25,5,3].

Interestingly, Justh et al. [17] have shown that cyclic pursuit on the plane in which the agents have constant speed and limited steering abilities converge to a steady state in which the agents follow a circle on the plane. Using a slightly different model, Marshall et al. [24] proved that the equilibrium formations are uniform circles and the radius can be determined by a designable parameter. In both cases convergence is not guaranteed and the circle location is dependent on the agents' initial location.

Related formation problems on ring graphs include the tasks of gathering, exploration and perpetual exploration (patrolling). The gathering (or rendezvous) goal is achieved when all agents are located at a single node of the ring. Flocchini et al. [12] considered gathering under limited visibility i.e. every agent can sense only the agents in his current location. In their model, either n or k are known to the agents and the agents can place tokens at nodes. Dessmark et al. [7] and later Kowalski et al. [21] studied the problem of gathering two robots in general graphs, trees and rings. Synchronous agents with distinct identifiers were considered where the identifiers were used in order to break symmetry. Later, De Marco et al. [23] considered the same problem under an asynchronous model but allowing the agents to meet on edges (and not only on vertices). Gasieniec et al. [14] addressed gathering of any number of agents on oriented rings assuming limited visibility. Klasing et al. proposed two complementary approaches for gathering: using symmetry breaking [20] or by symmetry preserving [19]. Note that in [7,21,23,20,19], the assumption was that every agent can sense the whole ring.

In the ring exploration task, the agents are required to visit every node of the ring and to stop afterwards. This task is challenging because the ring is unoriented and the agents cannot "remember" the direction of their last move (otherwise, they could simply continue forward). Considering deterministic algorithms, Flocchini et al. [11] have shown that the problem is unsolvable if n and k are co-prime. They have also provided an algorithm which solves the problem otherwise. In their model it is assumed that many agents can be in one node. A similar model was later considered by Lamani et al. [22]. By introducing probabilistic algorithms, Devismes et al. [8] were been able to remove the co-prime constraint.

Baldoni et al. [1] considered the task of perpetual exploration in which every agent is required to visit all the vertices of the graph infinitely many times. In their model, every vertex or edge may be occupied by at most one agent at a time. They examined the maximum number of agents that can simultaneously perform perpetual exploration. Blin et al. [2] have proposed an algorithm by which the agents perpetually explore an unoriented ring. While following the proposed algorithm, the agents "remember" the direction of their last move by maintaining an asymmetric formation.

To the best of our knowledge, we are the first to consider the discrete case of the problem of balanced deployment on a ring environment (i.e. spreading over a ring graph) under limited visibility. Furthermore, in previous work regarding uniform spread over a continuous ring (or a line) the agent speed was unlimited i.e. the maximum step size an agent could take in a single time cycle was unbounded so the time bounds previously achieved were proportional only to the number of agents. In our work, we limit the agent speed to one edge per time cycle so the time bounds achieved are relative to the ring size as well.

3. Preliminaries

Denote the set of agents by A where $|A| = k$. Whenever we fix an agent a, let a_{+1} (a_{-1}) be the clockwise (counter-clockwise) neighbor of agent a on the ring graph. Similarly a_{+2} will denote the clockwise neighbor of a_{+1} and so on. A step taken by an agent clockwise is a "forward step" and a counter-clockwise step is a "backward step". Let $d(a, b)$ be the distance between agents a and b, defined as the number of edges on the clockwise path from a to b. Note that $d(a, b) + d(b, a) = n$. $d_{-1}(a)$ is the distance between a_{-1} and a; $d_{+1}(a)$ is the distance between a and a_{+1}. The set $A[a, b) \subseteq A$ includes, by definition, all the agents between a and b including a but not including b. Note that $A[a, b) + A[b, a) = A$. An illustration of these notations can be found in Fig. 1.

Recall that V is the agents' visibility range. If $d(a, a_{+1}) \le V$ then agents a and a_{+1} can measure $d(a, a_{+1})$. If $d(a, a_{+1}) > V$ the agents cannot sense each other and will use $d(a, a_{+1}) = \infty$. We shall further assume that every agent can measure only the distance to his two neighbors and cannot "see beyond them" e.g. even in the case where $d(a, a_{+2}) \le V$, agent a cannot measure the distance to a_{+2}.

Throughout the paper mathematical operations on agent indices are modulo k. When we explicitly add t to the indices of a quantity we refer to the value of that quantity at time t, e.g. $d(a, b; t)$ is the distance between agents a and b at time t.

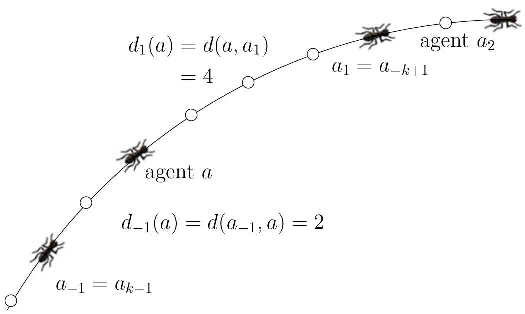

Fig. 1. Notations.

4. Uniform spread

4.1. Impossibility results

For the case of unoriented rings i.e. when the agents have no access to a common orientation of the ring, *uniform spread* cannot be achieved by a uniform deterministic algorithm. This holds even if the the agents have memory, global knowledge and unlimited sensing abilities. Like many other impossibility results in distributed systems, the swarm cannot achieve *uniform spread* simply because the agents are unable to break symmetry. Our impossibility result is the discrete variant of a similar result by Flocchini et al. [13]. The result we obtain is stronger since it holds under both SM and SSM where the continuous space result holds only for the SSM.

Theorem 1. *There is no uniform deterministic algorithm that achieves a* uniform spread *on unoriented rings.*

Proof. We will prove the theorem under SM. Consider the case of an even number of agents k and a ring of size $n = m \cdot k$ where $m \geq 2$ is an integer. Fix an agent a, and consider a configuration $C(l)$ defined as follows: for every i such that $0 \leq i \leq k/2 - 1$,

$$d_{+1}(a_{2i}) = d_{-1}(a_{2i+1}) = 2l + 1$$
$$d_{-1}(a_{2i}) = d_{+1}(a_{2i-1}) = 2(m - l) - 1$$

where l is an integer such that $0 \leq l \leq m$. Note that configuration $C(l)$ is not balanced for any l.

Let A_{even} be the group of agents a_{+2i} where $0 \leq i \leq k/2 - 1$ and A_{odd} — the agents a_{+2i+1}. All agents of the set A_{even} sense the same world view. All agents of the set A_{odd} sense the same view. Furthermore, the view of the agents of A_{odd} is a "mirror image" of the view of the agents of A_{even} e.g. d_{+1} (resp. d_{-1}) of any agent of A_{even} equals d_{-1} (resp. d_{+1}) of any agent of A_{odd}. Because the algorithm is uniform and deterministic, when an agent of A_{even} takes a clockwise step, all other agents of A_{even} take a clockwise step and all agents of A_{odd} take a counter-clockwise step. The resulting configuration will again be a $C(l)$-type configuration (with a different l value) hence not balanced. This shows that there are always "initial" unbalanced class of configurations from which the agents applying a uniform deterministic algorithm will never escape. □

In contrast to the result above, on oriented rings, *uniform spread* is possible. We shall also ask: "what is the minimal visibility range that enables *uniform spread*?". Theorem 2 below states that *uniform spread* is impossible if the agents' sensing range is strictly less than $\lfloor n/k \rfloor$. Algorithm 1, presented in Section 4.2, converges to a balanced configuration for $V \geq \lfloor n/k \rfloor$ so the question is completely settled (the convergence proof for $V \geq \lfloor n/k \rfloor$ can be found in Appendix B of our TR [10]).

Theorem 2. *On an oriented ring, if the sensing range of the agents is strictly smaller than $\lfloor n/k \rfloor$, no uniform deterministic algorithm can achieve uniform spread.*

Proof. We shall again prove the theorem under SM. Consider $k \geq 3$ agents $a_{+1} \cdots a_{+k}$ on a ring of size $n = m \cdot k - 1$ where $m \geq 2$ is an integer. Let the sensing range of the agents (V) be strictly smaller than $\lfloor n/k \rfloor$. Consider the non-balanced configuration in which $d(a_{+1}, a_{+2}) = n - \lfloor n/k \rfloor (k - 1) > \lceil n/k \rceil$, all other distances between adjacent agents being $\lfloor n/k \rfloor$. Here all the gaps are strictly greater than V so the agents cannot sense each other. For every agent $d_{-1} = d_{+1} = \infty$. Since the algorithm is *oblivious,* all agents will take the same actions, henceforth the configuration will remain unchanged. □

4.2. Uniform spread on an oriented ring

In order to bypass the impossibility result regarding unoriented rings we discuss oriented rings i.e. we assume that the agents agree on a common orientation of the ring (alternately, in order to break symmetry, one could consider probabilistic algorithms or use a different synchronicity model, see e.g. [15]). The *uniform spread* algorithm we propose is very simple. Every agent tries to balance the two distances to its nearest neighbor ahead (d_{+1}) and behind (d_{-1}). In order to break symmetry, we do not allow the agents to take forward steps. If $d_{-1} > d_{+1}$ the agent takes a backward step hence decreasing d_{-1} and increasing d_{+1}. If $d_{+1} > d_{-1}$ the agent will not move while other agents will act toward balancing the system.

The proposed algorithm is presented as Algorithm 1. A similar algorithm for the continuous space problem can be found in Section 5.1 of [13]. The algorithm correctness and time complexity are discussed below. We show that the algorithm converges to a balanced configuration in $O(n)$ *rounds* if $V \geq \lceil n/k \rceil$. In case all agents start from the same vertex, a balanced configuration cannot be achieved faster than $\Omega(n)$ so the algorithm convergence time is optimal.

Algorithm 1: Uniform Spread

```
1 if d_{-1} > d_{+1} then
2     Take a step backward.
```

The functions f, g are defined by

$$f(x) \triangleq x \cdot \lceil n/k \rceil$$
$$g(x) \triangleq x \cdot \lfloor n/k \rfloor .$$

We show in Lemma 4 that $f(j-i)$ and $g(j-i)$ are respectively an upper and lower bound on the distance between any two agents a_{+i}, a_{+j} in a balanced configuration.

Lemma 4. *In a balanced configuration for every two agents a_{+i}, a_{+j}*

$$g(j-i) \leq d\left(a_{+i}, a_{+j}\right) \leq f(j-i) .$$

Proof. In a balanced configuration the distance between any two consecutive agents is at most $\lceil n/k \rceil$. So

$$d\left(a_{+i}, a_{+j}\right) = \sum_{l=i}^{l=j-1} d_{+1}(a_{+l}) \leq (j-i) \cdot \lceil n/k \rceil = f(j-i) .$$

The lower bound is proved similarly. □

For any two agents a_{+i}, a_{+j} let

$$u\left(a_{+i}, a_{+j}\right) \triangleq d\left(a_{+i}, a_{+j}\right) - f(j-i)$$
$$l\left(a_{+i}, a_{+j}\right) \triangleq g(j-i) - d\left(a_{+i}, a_{+j}\right) .$$

In case $u\left(a_{+i}, a_{+j}\right) > 0$, the agents a_{+i}, a_{+j} are too far apart. So the distance $d\left(a_{+i}, a_{+j}\right)$ must be reduced by at least $u\left(a_{+i}, a_{+j}\right)$ edges before the configuration is balanced. Since forward steps are not allowed, a_{+j} must take at least $u\left(a_{+i}, a_{+j}\right)$ backward steps in order to reach a balanced configuration. Similarly, in case $l\left(a_{+i}, a_{+j}\right) > 0$, the agents are too close so the distance $d\left(a_{+i}, a_{+j}\right)$ must be increased by a_{+i}.
Define the upper and lower loads on agent a by

$$u(a) \triangleq \max_j \left\{u\left(a_{+j}, a\right)\right\}$$
$$l(a) \triangleq \max_j \left\{l\left(a, a_{+j}\right)\right\} .$$

Intuitively, before forming a balanced configuration, agent a must take at least $\max\{u(a), l(a)\}$ steps. Next, two technical lemmas regarding the upper and lower loads are presented.

Lemma 5. *Let a_{-i} be the agent such that $u(a_{-i}, a) = u(a)$ and let a_{+j} be the agent such that $l\left(a, a_{+j}\right) = l(a)$ then*

1. $d_{+1}(a_{-i}) \geq \lceil n/k \rceil$.
2. $d_{-1}(a_{-i}) \leq d_{+1}(a_{-i})$.
3. $d\left(a_{j-1}, a_{+j}\right) \leq \lfloor n/k \rfloor$.
4. $d_{+1}\left(a_{+j}\right) \geq d_{-1}\left(a_{+j}\right)$.

Proof. To prove item 1, assume toward contradiction that $d_{+1}(a_{-i}) < \lceil n/k \rceil$. So

$$\begin{aligned} u(a) &= u(a_{-i}, a) = u(a_{-i+1}, a) + d_{+1}(a_{-i}) - \lceil n/k \rceil \\ &< u(a_{-i+1}, a) \leq u(a) \end{aligned}$$

a contradiction.

To prove item 2, assume toward contradiction that $d_{-1}(a_{-i}) > d_{+1}(a_{-i})$ so

$$\begin{aligned} u(a) &\geq u(a_{-i-1}, a) = d(a_{-i-1}, a) - f(i+1) \\ &= u(a_{-i}, a) + d_{-1}(a_{-i}) - \lceil n/k \rceil \\ &> u(a_{-i}, a) + d_{+1}(a_{-i}) - \lceil n/k \rceil \\ &\geq u(a_{-i}, a) = u(a) \end{aligned}$$

a contradiction. The last inequality results from item 1. Items 3 and 4 are proved similarly. □

Lemma 6. *There is an agent a for which $u(a) = 0$ and there is an agent b for which $l(b) = 0$.*

Proof. We prove the lemma for $u(a)$, the proof for $l(b)$ is similar. Assume (toward a contradiction) that for any agent a, $u(a) \geq 0$ i.e. there is an agent a' such that $u(a', a) > 0$. Fix any agent a^0. Let a^1 be an agent such that $u(a^1, a^0) > 0$. Choose a^2 such that $u(a^2, a^1) > 0$ and so on. Using this process we create a chain of $k+1$ agents in which at least one agent appears twice. Denote this agent by a^0 and observe the sub-chain $a^0 a^1 \cdots a^m a^0$. This chain of agents circles the ring an integer number, p times hence

$$\sum_{i=0}^{m} d(a^{i+1}, a^i) = p \cdot n$$

$$\sum_{i=0}^{m} \left|A[a^{i+1}, a^i)\right| = p \cdot k.$$

Using the linearity of f we have

$$\begin{aligned} \sum_{i=0}^{m} u(a^{i+1}, a^i) &= \sum_{i=0}^{m} \left[d(a^{i+1}, a^i) - f(\left|A[a^{i+1}, a^i)\right|)\right] \\ &= p \cdot n - \lceil n/k \rceil \cdot \sum_{i=0}^{m} \left|A[a^{i+1}, a^i)\right| \\ &\leq p \cdot n - \lceil n/k \rceil \cdot pk \leq 0. \end{aligned}$$

However $u(a^{i+1}, a^i)$ is strictly positive for any i so we also have

$$\sum_{i=0}^{m} u(a^{i+1}, a^i) > 0$$

a contradiction. Therefore there is an agent a for which $u(a) = 0$. □

For every agent a let $u_0(a)$ be the agent such that $u(u_0(a)) = 0$ and the cardinality of the set $A[a, u_0(a))$ is minimal. In case $u(a) = 0$ then $u_0(a) = a$ and $|A[a, u_0(a))| = 0$. Similarly, let $l_0(a)$ be the agent such that $l(l_0(a)) = 0$ and the cardinality of the set $A[a, l_0(a))$ is minimal. Such agents exist by Lemma 6. Define the upper and lower potentials of agent a by

$$v_u(a) \triangleq 2u(a) + |A[a, u_0(a))|$$
$$v_l(a) \triangleq 2l(a) + |A[a, l_0(a))|.$$

The system upper and lower potentials are given by

$$V_u \triangleq \max_{a \in A} \{v_u(a)\}$$
$$V_l \triangleq \max_{a \in A} \{v_l(a)\}.$$

By Lemma 4, the potential of a balanced configuration is zero. On the other hand, if $V_u = V_l = 0$ then the system is in a balanced configuration. We show in the next lemma that the upper and lower potentials of an agent do not increase.

Lemma 7. *Fix an agent a. Then $u(a)$, $v_u(a)$, $l(a)$ and $v_l(a)$ are non-increasing under SSM model.*

Proof. We prove the lemma for $u(a)$ and $v_u(a)$. The proofs for $l(a)$ and $v_l(a)$ are similar. Assume toward contradiction that $u(a)$ has increased at time t i.e. $u(a; t+1) > u(a; t)$. Let a_{-i} be an agent such that $u(a_{-i}, a; t+1) = u(a; t+1)$. We can write

$$u(a_{-i}, a; t+1) = u(a; t+1) > u(a; t) \geq u(a_{-i}, a; t).$$

So $u(a_{-i}, a)$ has increased at time t. $u(a_{-i}, a)$ increases when $d(a_{-i}, a)$ increases i.e. when agent a_{-i} takes a step backward. Hence at time t agent a_{-i} was active and according to the algorithm,

$$d_{-1}(a_{-i}; t) > d_{+1}(a_{-i}; t)$$

in contradiction with Lemma 5(2).

We have shown that for every agent a, $u(a)$ does not increase. In particular, it holds for agent $u_0(a)$. So $u(u_0(a))$ remains zero and $|A[a, u_0(a))|$ does not increase hence $v_u(a)$ does not increase. □

Recall that a *round* is a time period in which every agent was active at least once. The next two lemmas show that in every *round* both the upper and lower system potentials decrease. All the claims stated so far hold for any V. The next two lemmas hold for $V \geq \lceil n/k \rceil$ and $V \geq \lfloor n/k \rfloor$ respectively.

Lemma 8. *If $V_u(t_1) > 0$, $V \geq \lceil n/k \rceil$, and between times t_1 and t_2 all the agents were active at least once, $V_u(t_2) < V_u(t_1)$.*

Proof. Let a be an agent such that $v_u(a; t_1) = V_u(t_1)$ and let t_a be the first time cycle in which a is active after t_1. We will prove the lemma by showing that

$$u(a; t_2) \leq u(a; t_a + 1) < u(a; t_a) \leq u(a; t_1)$$

hence

$$v_u(a; t_2) < v_u(a; t_1). \tag{1}$$

Eq. (1) holds for any agent a such that $v_u(a; t_1) = V_u(t_1)$. Because v_u is non increasing for any agent, we conclude that $V_u(t_2) < V_u(t_1)$.

To prove Eq. (1), let a be an agent such that $v_u(a; t_1) = V_u(t_1)$ and let a_{-i} be an agent such that $u(a_{-i}, a; t_1) = u(a; t_1)$. Then, for any $t_1 \leq t \leq t_a$, the following hold:

1. $u(a_{-i}, a; t) = u(a; t)$ and agent a_{-i} will not take a step backward at time t.
   Since a is not active before time t, $u(a_{-i}, a; t)$ can only be changed if a_{-i} takes a step. According to Lemma 5(2), as long as $u(a_{-i}, a; t) = u(a; t)$, agent a_{-i} will not move.
2. $d_{-1}(a; t) > d_{+1}(a; t)$.
   Assume toward contradiction that $u(a_{-1}; t) \geq u(a; t)$. Since $|A[a_{-1}, u_0(a_{-1}))| > |A[a_{-1}, u_0(a))|$, $v_u(a_{-1}; t) > v_u(a; t)$ in contradiction to $v_u(a; t)$ maximality. So $u(a_{-1}; t) \leq u(a; t) - 1$ and

$$\begin{aligned} d(a_{-i}, a_{-1}; t) - f(i-1) &= u(a_{-i}, a_{-1}; t) \leq u(a_{-1}; t) \leq u(a; t) - 1 \\ &= u(a_{-i}, a; t) = d(a_{-i}, a; t) - f(i) - 1 \\ d(a_{-1}, a; t) &\geq f(i) - f(i-1) + 1 = \lceil n/k \rceil + 1. \end{aligned} \tag{2}$$

   By a similar argument on a_{+1} we get

$$\begin{aligned} d(a_{-i}, a_{+1}; t) - f(i+1) &= u(a_{-i}, a_{+1}; t) \leq u(a_{+1}; t) \leq u(a; t) \\ &= u(a_{-i}, a; t) = d(a_{-i}, a; t) - f(i) \\ d(a, a_{+1}; t) &\leq f(i+1) - f(i) = \lceil n/k \rceil. \end{aligned} \tag{3}$$

   By summing Eqs. (2), (3) we get

$$d_{-1}(a; t) - d_{+1}(a; t) = d(a_{-1}, a; t) - d(a, a_{+1}; t) \geq 1.$$

3. $d_{+1}(a; t) \leq V$.
   Using Eq. (3),

$$d_{+1}(a; t) = d(a, a_{+1}; t) \leq \lceil n/k \rceil \leq V.$$

By Item 1, agent a terms for taking a backward step are fulfilled and by Item 3 the agent can sense that. So agent a will take a step backward at time t_a while a_{-i} will not move (Item 1) hence $v_1(a; t_a + 1) < v_1(a; t_a)$. □

Lemma 9. *If $V_l(t_1) > 0$, $V \geq \lfloor n/k \rfloor$, and between times t_1 and t_2 all the agents were active at least once, $V_l(t_2) < V_l(t_1)$.*

Proof. The proof is similar to the proof of Lemma 8. Let a be an agent such that $v_l(a; t_1) = V_l(t_1)$ and let t_a be the first time cycle in which a is active after t_1. We will prove the lemma by showing

$$\begin{aligned} l(a; t_2) &\leq l(a; t_a + 1) < l(a; t_a) \leq l(a; t_1) \\ v_l(a; t_2) &< v_l(a; t_1). \end{aligned}$$

Since the argument holds for any agent a such that $v_l(a; t_1) = V_l(t_1)$ and using Lemma 7, $V_l(t_2) < V_l(t_1)$.

Fix an agent a such that $v_l(a; t_1) = V_l(t_1)$ and let a_{+i} be the agent such that $l(a, a_{+i}; t_1) = l(a, a_{+i}; t_1)$. For any $t_1 \le t \le t_a$, the following hold:

1. $l(a, a_{+i}; t) = l(a; t)$ and agent a_{+i} will not take a step backward at time t.
   Since a is not active before time t, $l(a, a_{+i}; t)$ can only be changed if a_{+i} takes a step backward. According to Lemma 54, as long as $l(a, a_{+i}; t) = l(a; t)$, agent a_{+i} will not move.
2. $d_{-1}(a; t) > d_{+1}(a; t)$.
   Assume toward contradiction that $l(a_{-1}; t) \ge l(a; t)$. Since $\left|A_{a_{-1}, l_0(a_{-1})}\right| > \left|A_{a, l_0(a)}\right|$, $v_l(a_{-1}; t) > v_l(a; t)$ in contradiction to $v_l(a; t)$ maximality. So $l(a_{-1}; t) \le l(a; t) - 1$ and

$$\begin{aligned} g(i+1) - d(a_{-1}, a_{+i}; t) &= l(a_{-1}, a_{+i}; t) \le l(a_{-1}; t) \le l(a; t) - 1 \\ &= l(a, a_{+i}; t) - 1 = g(i) - d(a, a_{+i}; t) - 1 \\ d(a_{-1}, a; t) &\ge g(i+1) - g(i) + 1 = \lfloor n/k \rfloor + 1. \end{aligned} \tag{4}$$

   By a similar argument on a_{+1} we get $l(a_{+1}; t) \le l(a; t)$ so

$$\begin{aligned} g(i-1) - d(a_{+1}, a_{+i}; t) &= l(a_{+1}, a_{+i}; t) \le l(a_{+1}; t) \le l(a; t) \\ &= l(a, a_{+i}; t) \le g(i) - d(a, a_{+i}; t) \\ d(a, a_{+1}; t) &\le g(i) - g(i-1) = \lfloor n/k \rfloor . \end{aligned} \tag{5}$$

   Summing Eqs. (4), (5) we get

$$d_{-1}(a; t) - d_{+1}(a; t) = d(a_{-1}, a; t) - d(a, a_{+1}; t) = 1.$$

3. $d_{+1}(a; t) \le V$.
   Using Eq. (5),

$$d_{+1}(a; t) = d(a, a_{+1}; t) \le \lfloor n/k \rfloor \le V.$$

By Item 2, agent a terms for taking a backward step are fulfilled and by Item 3 the agent can sense that. So agent a will take a step backward at time t_a while a_{+i} will not move (Item 1) hence $v_l(a; t_a + 1) < v_l(a; t_a)$. □

In order to bound the convergence time, we first bound the potential of the initial configuration.

Lemma 10. *For any configuration, $V_u \le 2n + k$ and $V_l \le 2n + k$.*

Proof. Regarding V_u, the largest distance between any two agents is n. The cardinality of any set of agents is at most k. Hence

$$\begin{aligned} v_u(a) &\le 2[d(b, a) - f(i)] + |A[a, u_0(a))| \le 2(n - \lceil n/k \rceil) + k \\ &\le 2n + k. \end{aligned}$$

Regarding V_l, the minimal distance between any two agents is zero so

$$\begin{aligned} v_l(a) &\le 2[g(i) - d(b, a)] + |A[a, l_0(a))| \le 2(\lfloor n/k \rfloor \cdot k - 0) + k \\ &\le 2n + k. \quad \square \end{aligned}$$

Theorem 3. *A swarm of agents following Algorithm 1 with $V \ge \lceil n/k \rceil$ will form a balanced configuration within $2n + k$ rounds under SSM and $2n + k$ time cycles under SM.*

Proof. We will first prove the theorem under SSM. According to Lemma 10, $V_u(t = 0) \le 2n + k$. By Lemma 8, every *round*, V_u decreases by at least one. Let t' be the time after $2n + k$ *rounds*. Then

$$V_u\left(t'\right) \le V_u(t = 0) - (2n + k) \le 0.$$

The same holds for V_l. So at time t', $V_u\left(t'\right) = V_l\left(t'\right) = 0$ thus the configuration is balanced. In SM every *round* takes one time cycle so a balanced configuration is achieved within t' time cycles. □

5. Quiescent uniform spread

5.1. Impossibility result

The next theorem proves that achieving a quiescent balanced configuration is impossible under our *oblivious* model in which every agent can sense only the distance to his two neighbors. The result holds for any V. The impossibility result does not hold under a stronger model in which every agent can sense all agents in his visibility range. This stronger model is beyond the scope of this paper and will be analyzed elsewhere.

Theorem 4. *If every agent can sense the distance to only his two neighbors on the ring, there is no* oblivious *deterministic algorithm that achieves a* quiescent uniform spread.

Proof. Assume toward contradiction that a swarm of agents following an *oblivious* algorithm A form a quiescent balanced configuration. Fix the swarm size (k) and consider a ring of n vertices where n/k is an integer. There is a single balanced configuration in which for every agent $d_{+1} = d_{-1} = n/k$. So in this configuration, all agents are quiescent. Consider a ring of n vertices where n/k is a fraction. In every balanced configuration there is an agent a such that $d_{+1}(a) = \lceil n/k \rceil$, $d_{-1}(a) = \lfloor n/k \rfloor$ and there is an agent b such that $d_{+1}(b) = \lfloor n/k \rfloor$, $d_{-1}(b) = \lceil n/k \rceil$. Since one of the balanced configuration must be quiescent, agents a and b do not move. Algorithm A is *oblivious* so all agents run the same algorithm. We conclude that every agent such that $|d_{+1} - d_{-1}| \leq 1$ does not move under algorithm A.

To construct the counter example fix $k \geq 4$ and let $n = m \cdot k + 4$ where m is a positive integer. Assume the following initial configuration:

$$\begin{aligned} d(a_{+1}, a_{+2}) &= m + 1 \\ d(a_{+2}, a_{+3}) &= m + 2 \\ d(a_{+3}, a_{+4}) &= m + 1 \end{aligned}$$

and all other distances equal m. For every agent $|d_{+1} - d_{-1}| \leq 1$ so all agents do not move and the configuration remains unchanged indefinitely. However, the configuration is not balanced, a contradiction. □

5.2. Quiescent semi-stable configuration

In the previous section we have shown that forming a quiescent balanced configuration is impossible. The impossibility result is based on the observation that under any algorithm, every agent for which $|d_{+1} - d_{-1}| \leq 1$ must not move. Based on that observation we present Algorithm 2 in which every agent is quiescent as long as

$$d_{-1} - d_{+1} \leq 1. \tag{6}$$

We show in Lemma 11 that when Eq. (6) holds for all agents, they form a semi-balanced configuration (recall Definition 3 from Section 1). The algorithm requires a slightly larger visibility range i.e. $V \geq \frac{n}{k} + \frac{k}{2}$. Note that in case $n/k \gg k$, a semi-balanced configuration is (almost) balanced. The algorithm correctness and time complexity are discussed below.

Algorithm 2: Quiescent Stable Configuration

```
if d_{-1} > d_{+1} + 1 then
    Take a step backward.
```

Lemma 11. *If Eq. (6) holds for all agents then the agents form a semi-balanced configuration.*

Proof. We shall show only the upper bound, the lower bound is proved similarly. Assume toward contradiction that there is a gap larger then $\frac{n}{k} + \frac{k}{2}$. Let a be the agent such that $d_{-1}(a) > \frac{n}{k} + \frac{k}{2}$. By implying Eq. (6) we get

$$\begin{aligned} d_{-1}(a_{+1}) &= d_{+1}(a) \geq d_{-1}(a) - 1 > \frac{n}{k} + \frac{k}{2} - 1 \\ d_{-1}(a_{+2}) &= d_{+1}(a_{+1}) \geq d_{-1}(a_{+1}) - 1 > \frac{n}{k} + \frac{k}{2} - 2 \\ &\vdots \\ d_{-1}(a_{+i}) &> \frac{n}{k} + \frac{k}{2} - i. \end{aligned}$$

So

$$n = \sum_{i=0}^{k-1} d_{-1}(a_{+i}) > \sum_{i=0}^{k-1} \left[\frac{n}{k} + \frac{k}{2} - i \right] = n + \frac{k^2}{2} - \frac{k(k-1)}{2} = n + \frac{1}{2}$$

a contradiction. □

The convergence proof of Algorithm 2 resembles the proof of Algorithm 1. The proof is based on potential functions which are similar to the ones used in the proof of Algorithm 1.

The function $h(i, d)$ is defined by

$$h(i, d) \triangleq i \cdot d + \sum_{j=1}^{i} j = i \cdot d + \frac{i(i+1)}{2}.$$

For every agent a let

$$s(a_{-i}, a) \triangleq d(a_{-i}, a) - h(i, d_{+1}(a))$$
$$s(a) \triangleq \max_i \{s(a_{-i}, a)\}.$$

The intuition behind the definition of s is the following: if $s(a) \leq 0$ then Eq. (6) holds for agent a. Therefore, by Lemma 11, if for all agents $s = 0$ then they form a semi-balanced configuration. Define the potentials of an agent and of a system respectively via:

$$v_s(a) \triangleq \begin{cases} 2 \cdot s(a) + d_{+1}(a) & s(a) > 0 \\ 0 & \text{else} \end{cases}$$
$$V_s \triangleq \max_{a \in A} \{v_s(a)\}.$$

When $V_s = 0$, the agents form a semi-balanced configuration. The next few lemmas and theorem show that V_s will be zeroed within $3n$ *rounds*. We first present two technical lemmas regarding s and v_s.

Lemma 12. *Let a_{-i} be an agent such that $s(a) = s(a_{-i}, a)$ then*

1. $d_{+1}(a_{-i}) \geq h(i, d_{+1}(a)) - h(i-1, d_{+1}(a))$.
2. $d_{-1}(a_{-i}) \leq d_{+1}(a_{-i}) + 1$.

Proof. To prove item 1, assume toward contradiction that $d_{+1}(a_{-i}) < h(i, d_{+1}(a)) - h(i-1, d_{+1}(a))$ then

$$\begin{aligned} s(a) &= s(a_{-i}, a) = d(a_{-i}, a) - h(i, d_{+1}(a)) \\ &= s(a_{-i+1}, a) + d_{+1}(a_{-i}) - [h(i, d_{+1}(a)) - h(i-1, d_{+1}(a))] \\ &< s(a) \end{aligned}$$

a contradiction.

To prove 2, assume toward contradiction that $d_{-1}(a_{-i}) > d_{+1}(a_{-i}) + 1$ then using item 1,

$$\begin{aligned} s(a) &\geq s(a_{-i-1}, a) = d(a_{-i-1}, a) - h(i+1, d_{+1}(a)) \\ &= d(a_{-i}, a) - h(i, d_{+1}(a)) + h(i, d_{+1}(a)) + d_{-1}(a_{-i}) - h(i+1, d_{+1}(a)) \\ &> s(a_{-i}, a) + d_{+1}(a_{-i}) + 1 + h(i, d_{+1}(a)) - h(i+1, d_{+1}(a)) \\ &\geq s(a) + 2h(i, d_{+1}(a)) - h(i+1, d_{+1}(a)) - h(i-1, d_{+1}(a)) + 1 \\ &= s(a) \end{aligned}$$

a contradiction. □

Lemma 13. *Let a be an agent such that $V(s) = v_s(a) > 0$ then,*

1. $d_{-1}(a) \geq d_{+1}(a) + 2$.
2. $d_{-1}(a_{+1}) \leq d_{+1}(a_{+1}) + 1$.
3. $d_{+1}(a) \leq \frac{n}{k} + \frac{k}{2}$.

Proof. Let a_{-i} be an agent such that $s(a) = s(a_{-i}, a)$.

1. Assume toward contradiction that $d_{-1}(a) \leq d_{+1}(a) + 1$, then

$$\begin{aligned} v_s(a) &= 2d(a_{-i}, a) - 2h(i, d_{+1}(a)) + d_{+1}(a) \\ &= 2d(a_{-i}, a_{-1}) - 2h(i-1, d_{+1}(a_{-1})) + 2d_{+1}(a_{-1}) \\ &\quad + 2h(i-1, d_{+1}(a_{-1})) - 2h(i, d_{+1}(a)) + d_{+1}(a) \\ &\leq v_s(a_{-1}) + d_{+1}(a_{-1}) + 2(i-1)d_{+1}(a_{-1}) + i(i-1) \\ &\quad - 2i \cdot d_{+1}(a) - i(i+1) + d_{+1}(a) \\ &= v_s(a_{-1}) + (2i-1)d_{-1}(a) - (2i-1)d_{+1}(a) - 2i \\ &\leq v_s(a_{-1}) - 1 < v_s(a_{-1}) \end{aligned}$$

in contradiction with $v_s(a)$ maximality.

2. Assume toward contradiction that $d_{-1}(a_{+1}) \geq d_{+1}(a_{+1}) + 2$.

$$\begin{aligned} v_s(a_{+1}) &\geq 2s(a_{-i}, a_{+1}) + d_{+1}(a_{+1}) = 2d(a_{-i}, a_{+1}) - 2h(i+1, d_{+1}(a_{+1})) + d_{+1}(a_{+1}) \\ &= v_s(a) + (2i+1)[d_{-1}(a_{+1}) - d_{+1}(a_{+1})] + i(i+1) - (i+1)(i+2) \\ &\geq v_s(a) + 2(2i+1) - 2(i+1) > v_s(a) \end{aligned}$$

in contradiction with $v_s(a)$ maximality.

3. Assume toward contradiction that $d_{+1}(a) \geq \frac{n}{k}+\frac{k}{2}+1$. We will first show, by induction on j, that for every $0 \leq j \leq k-1-i$, $d_{+1}\left(a_{+j}\right) \geq \frac{n}{k}+\frac{k}{2}+1-j$. For $j=0$ the claim holds trivially. Assume toward contradiction that the claim holds for $j-1$ but not for j then,

$$d_{+1}\left(a_{+j}\right) \leq \frac{n}{k}+\frac{k}{2}-j$$

using the induction hypothesis we write

$$\begin{aligned} d\left(a, a_{+j}\right) &= \sum_{i=0}^{j-1} d_{+1}\left(a_{+i}\right) \geq \sum_{i=0}^{j-1}\left[\frac{n}{k}+\frac{k}{2}+1-i\right] \\ &= j\left(\frac{n}{k}+\frac{k}{2}+\frac{(3-j)}{2}\right) \end{aligned}$$

finally,

$$\begin{aligned} v_s\left(a_{+j}\right) &\geq 2\left[d\left(a_{-i}, a_{+j}\right)-h\left(i+j, d_{+1}\left(a_{+j}\right)\right)\right]+d_{+1}\left(a_{+j}\right) \\ &= v_s(a)-d_{+1}(a)+2h\left(i, d_{+1}(a)\right)+2d\left(a, a_{+j}\right)-2h\left(i+j, d_{+1}\left(a_{+j}\right)\right)+d_{+1}\left(a_{+j}\right) \\ &> v_s(a) \end{aligned}$$

in contradiction with $v_s(a)$ maximality.

Using the claim we have just proved,

$$\begin{aligned} \sum_{j=0}^{k-i-1} d_{+1}\left(a_{+i}\right) &\geq \sum_{j=0}^{k-i-1}\left[\frac{n}{k}+\frac{k}{2}+1-j\right] \\ &= (k-i)\left(\frac{n}{k}+\frac{i+3}{2}\right) \end{aligned}$$

because $u(a)>0$,

$$\begin{aligned} d\left(a_{-i}, a\right) &\geq h\left(i, d_{+1}(a)\right)+1 \\ &\geq i\left(\frac{n}{k}+\frac{i+3}{2}\right)+i\frac{k}{2}+1 \end{aligned}$$

then

$$\begin{aligned} n &= d\left(a_{-i}, a_{k-i}\right)=d\left(a_{-i}, a\right)+\sum_{j=0}^{k-i-1} d_{+1}\left(a_{+i}\right) \\ &\geq n+ik+1+\frac{3k}{2}>n \end{aligned}$$

a contradiction. □

The next two lemmas show that V_s decreases with time. To be exact, in every *round*, the system potential decreases by at least one.

Lemma 14. *If $v_s(a;t)<V_s(t)$ then $v_s(a;t+1)<V_s(t)$ under SSM.*

Proof. Let a be an agent such that $v_s(a;t)<V_s(t)$ and assume toward contradiction that $v_s(a;t+1) \geq V_s(t)$. Let a_{-i} be an agent such that $s(a)=s\left(a_{-i}, a\right)$. By Lemma 12(2), a_{-i} cannot take a step so the distance $d\left(a_{-i}, a_l\right)$ cannot increase for any agent l. We are left with two cases:

Case 1. If $d_{+1}(a)$ has increased at time t, agent a has taken a step backward at time t which implies $d_{-1}(a;t)>d_{+1}(a;t)+1$ and $v_s(a;t)>0$. Let a_{-j} be the agent such that $v_s(a;t+1)=2s\left(a_{-j}, a;t+1\right)+d_{+1}(a;t+1)$. We can write

$$\begin{aligned} v_s(a;t+1) &= 2\left[d\left(a_{-j}, a;t+1\right)-h\left(j, d_{+1}(a;t+1)\right)\right]+d_{+1}(a;t+1) \\ &\leq 2\left[d\left(a_{-j}, a;t\right)-h\left(j, d_{+1}(a;t)\right)-j\right]+d_{+1}(a;t)+1 \leq v_s(a;t) \end{aligned}$$

in contradiction to the assumption that $v_s(a)$ increased at time t.

Case 2. If $d_{+1}(a)$ has decreased at time t, agent a_{+1} has taken a step backward at time t which implies $d_{-1}\left(a_{+1};t\right) \geq d_{+1}\left(a_{+1};t\right)+2$ and $v_s\left(a_{+1};t\right)>0$. Let a_{-j} be the agent such that $v_s(a;t+1)=2s\left(a_{-j}, a;t+1\right)+d_{+1}(a;t+1)$. Using

$$\begin{aligned} d_{+1}(a;t+1) &= d_{+1}(a;t)-1 \\ d_{+1}\left(a_{+1};t\right) &\leq d_{+1}(a;t)-2=d_{+1}(a;t+1)-1 \end{aligned}$$

we write

$$\begin{aligned} v_s(a_{+1};t) &\geq 2s\left(a_{-j},a_{+1};t\right)+d_{+1}(a_{+1};t) \\ &= v_s(a;t+1)+(2j+1)\left[d_{+1}(a;t+1)-d_{+1}(a_{+1};t)\right]-2j \\ &> v_s(a;t+1). \end{aligned}$$

So $V_s(t) \geq v_s(a_{+1};t) > v_s(a;t+1)$ in contradiction to the assumption that $v_s(a;t+1) > V_s(t)$. □

Lemma 15. *If $V_s(t_1) > 0$, $V \geq \frac{n}{k}+\frac{k}{2}$, and between times t_1 and t_2 all the agents were active at least once, $V_s(t_2) < V_s(t_1)$.*

Proof. Let a be an agent such that $v_s(a;t_1) = V_s(t_1)$ and let t_a be the first time cycle in which a is active after t_1. We will prove the lemma by showing

$$s(a;t_2) \leq s(a;t_a+1) < s(a;t_a) \leq s(a;t_1)$$

hence

$$v_s(a;t_2) < v_s(a;t_1). \quad (7)$$

Eq. (7) holds for any agent a such that $v_s(a;t_1) = V_s(t_1)$, using Lemma 14 we conclude that $V_s(t_2) < V_s(t_1)$.

Fix an agent a such that $v_s(a;t_1) = V_s(t_1)$ and let a_{-i} be the agent such that $s(a_{-i},a;t_1) = s(a;t_1)$. Then,

1. For any time t, $t_1 \leq t \leq t_a$ the following holds: (a) $s(a_{-i},a;t) = s(a;t)$; (b) agents a_{-i} and a_{+1} will not take a step at time t.

   Since a is not active before time t, if both a_{-i} and a_{+1} have not moved before time t then $s(a_{-i},a;t) = s(a;t)$. The proof is by induction on t. Assuming the claim holds up to time t, in particular $s(a_{-i},a;t) = s(a;t)$. According to Lemmas 12(2) and 13(2), both agents a_{-i} and a_{+1} will not take a step at time t. So $s(a_{-i},a;t+1) = s(a;t+1)$ which completes the induction step.
2. $d_{-1}(a;t_a) > d_{+1}(a;t_a)+1$.

   By the previous Item and Lemma 13(1).
3. $d_{+1}(a;t_a) \leq V$.

   By Item 1 and Lemma 13(3).

By Item 2, agent a terms for taking a step are fulfilled and by Item 3 the agent can sense that. So agent a will move backward at time t_a while a_{-i} will not move (Item 1) hence $s(a;t_a+1) < s(a;t_a)$. □

The next lemma and theorem complete the proof in the following manner: the potential of the initial configuration is bounded by $3n$ and in every *round* the potential decreases. Therefore within $3n$ rounds $V_s \leq 0$ and the agents reach a quiescent semi-balanced configuration.

Lemma 16. *For any configuration $V_s \leq 3n$.*

Proof. Using the trivial bound that the distance between any two agents is at most n we write,

$$\begin{aligned} s(a) &= d(a_{-i},a)-h(i,d_{+1}(a)) \leq n \\ v_s(a) &\leq 2\cdot s(a)+d_{+1}(a) \leq 3n. \end{aligned}$$ □

Theorem 5. *A group of k agents following Algorithm 2 where $V \geq \frac{n}{k}+\frac{k}{2}$ will form a quiescent semi-balanced configuration within $3n$* rounds *under SSM and $3n$ time cycles under SM.*

Proof. We shall prove the theorem under SSM. According to Lemma 16, $V_s(t=0) \leq 3n$. By Lemma 15, every *round* V_s decreases by at least one. Let t' be the time $3n$ *rounds* after the algorithm initialization. Then by time t'

$$V_s\left(t'\right) \leq V_s(t=0)-3n \leq 0.$$

By Lemma 11, at time t', the agents form a quiescent semi-balanced configuration. In SM every *round* takes one time cycle so a balanced configuration is achieved within t' time cycles. □

6. Conclusion

In this paper we have considered two variants of the problem of spreading a swarm of agents uniformly on a ring graph. In the first variant the agents are required to "dynamically" spread uniformly over the ring. We have shown that if the ring is unoriented or the sensing range of the agents is strictly less than $\lfloor n/k \rfloor$, this task is impossible. Considering an oriented ring and $V \geq \lceil n/k \rceil$, we have proposed an algorithm which achieves the task within optimal time. In the second variant, the agents are required to spread over the ring and stop once a balanced configuration is reached (quiescent spread). We have shown that under our model in which every agent can measure the distance only to his two neighbors, achieving this task is impossible. As a partial solution, we have proposed an algorithm which achieves a quiescent and "almost uniform" spread.

Some interesting open issues that remain to be answered are:

- In continuous space systems, the agent's speed is limited. Can our algorithms induce ϵ-*approximate* [13] algorithms for the limited speed continuous space case? If so, the time bounds achieved shall be with respect to the ring size, agent speed and the number of agents as opposed to state of the art bounds which consider the number of agents only.
- In case every agent can measure the distance to all agents in his visibility range, what is the minimal visibility range that enables a quiescent balanced configuration? Which algorithms achieve that?
- Under our model in which every agent can sense the distance to his two neighbors only, can the agents achieve a more uniform quiescent spread than the semi-balanced configuration we have presented? Perhaps a random algorithm can achieve better results on average?

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