The Resolution of Overlapping Echos

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Abstract—We present a new method for estimating the number and arrival times for overlapping signals with *a priori* known shape from noisy observations received by a sensor. The method is based on a recently developed eigenstructure technique for multitarget direction finding with passive antenna arrays and exploits the structure of the received signal covariance matrix. This problem is important in various applications such as radar and sonar data processing, geophysical/seismic exploration, and biomedical engineering. In many of these applications, a known signal is launched into a scattering medium and the returning response—in the form of delayed overlapping echos in noise—has to be processed to yield information on the nature and location of scatterers. The method presented also solves more general problems of signal detection and resolution.

I. Introduction

In many applications such as radar and sonar data processing, geological acoustic sounding, ultrasound-based nondestructive testing, and medical imaging procedures, one is faced with the problem of resolving an unknown number of closely spaced, overlapping, and noisy echos of a signal with *a priori* known shape. Several approaches for solving this problem have been studied so far. These include detection/deconvolution schemes, inverse filtering, least-squares, and maximum likelihood methods; see, e.g., [1]–[9]. The problem has many variations and is, in fact, a combined detection–estimation problem: one has to determine the number of returning echos and then apply an estimation procedure to determine their location in time.

Consider that a signal of known shape $s(\cdot)$ is sent through a medium that returns its echos from various locations. Over a window in time, beginning, say, at the moment the probing signal is emitted, a sensor receives a superposition of delayed and randomly scaled versions of $s(\cdot)$. The received signal can thus be written as

$$r(t) = \sum_{i=1}^{D} m_i\, s(t - \theta_i) + n(t) \tag{1}$$

where we assume that

1) the θ_i are the delay parameters, related to the location of the scattering objects

2) the m_i are random gains incorporating both scatterer characteristics and propagation fading through the medium, and

3) $n(\cdot)$ is an additive white noise process.

Note that in the above model, we assumed a baseband situation. This is done simply for convenience, and in the case of RF-modulated signals, we have a similar model for the received complex signal. In this circumstance, we would also have to deal with a random phase component multiplying the gains m_i, and the known shape of the probing pulse would simply be the envelope of the RF pulse. This and several other extensions will be discussed in more detail later, except for the remark that the randomness in amplitude (and phase) assumed in our method is more readily at hand in radar problems, arising from tiny variations in component echo delay that are small on the scale of estimation accuracy, but large enough to randomize the carrier phase of the echo. The significance of the randomness assumption will be evident in the analysis presented in Section II.

Suppose that the mode of probing is repetitive, i.e., at some predetermined rate, identically shaped pulses are launched into the medium. Assuming also that over the repetition period of, say, length T all returns have died out already, the succession of received signals may be considered as resulting from repeated independent experiments on the medium. Many conventional radar/sonar systems operate exactly in this way; in geological and other acoustic sounding applications, we would require a set of identical experiments to be performed.

Therefore, by assumption, we have access to an ensemble of K responses

$$\{r_j(t)\} = \left\{ \sum_{i=1}^{D} m_{ij}\, s(t - \theta_i) + n_j(t) \right\} \qquad j = 1, 2 \cdots K$$

in which the random components are results of independent trials. The problem we address is the following: given a set of responses as above, determine

1) the number of delayed echos present
2) the delays corresponding to each return
3) the second-order statistics of the gain parameters.

Our solution to this problem is based on a time-domain reinterpretation of an ingenious method developed and used in the context of multitarget direction finding with passive sensor arrays. We shall show that this so-called "multiple signal characterization" (or MUSIC) algorithm of Schmidt [10] and Bienvenu [11], which is used to determine the number of radiating sources and their location, can be used to provide the solution to a whole class of signal resolution problems of the type we are discussing. The direction finding algorithm cleverly exploits the

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eigenstructure of the covariance matrix of the received signals at the M sensors in the passive array. In a similar way, the signal resolution procedure relies on the eigenstructure of covariance matrix of M samples of the responses $r_j(\cdot)$. This is described in Section II.

In Section III, we discuss several extensions of the basic problem that can also be handled with the MUSIC approach, and Section IV presents a series of simulation results exhibiting the performance of the proposed signal resolution method.

II. Using Music to Resolve Overlapping Echos

The MUSIC Algorithm

Suppose an array of M sensors monitors the signals produced by D radiating sources. Assume further that the sensor array is in the far field of the sources so that only the radial direction of the sources is relevant in the pattern of received signals. (The waves generated by each source behave like plane waves coming from its direction.) Then the signals received by the M sensors can be modeled as follows [9]–[12]:

$$\begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_M(t) \end{bmatrix} = [A(\theta_1)\, A(\theta_2) \cdots A(\theta_D)] \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_D(t) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \\ \vdots \\ n_M(t) \end{bmatrix}$$

or, more compactly,

$$r(t) = As(t) + n(t) \tag{3}$$

where $r_i(\cdot)$ is the signal received at the ith sensor, $s_j(\cdot)$ is the signal generated by the jth source, $A(\theta)$ is the "signature" of a source in the direction θ, and n_i is an independent noise affecting the ith sensor.

The parametrized set of signature vectors $\{A(\theta)\}_{\theta \in \Theta}$ where Θ is the parameter set—usually $[0, 2\pi]$—is appropriately called the "array manifold" since it characterizes the directional properties of the sensor array. The parametrized array manifold may be obtainable in a closed analytical form (for simple spatial geometries like linear or circular arrays) or can be measured through field calibration procedures and then stored in a computer memory. It is usually assumed that, by careful choice of the array geometry, it has the following property: for any set of parameters θ with less than M elements, the array manifold vectors are linearly independent. This property ensures that there will be no ambiguities in the signal model (3) since a linear combination of two or more direction signatures will never provide the "phantom" signature of some different direction. This property is trivially satisfied by linear sensor arrays (for $\theta \in [0, \pi]$) since their array manifold has Vandermonde-type columns; see, e.g., [10], [12]. However, the significance of the MUSIC method is that it can apply to arbitrary arrays geometries, a feature that will be important in our applications. For a linear array, MUSIC is closely related to several previously known algorithms for spectral analysis and for direction finding (see, i.e., [9], [12], [17]–[19]). From (3), it is readily seen that the covariance of $r(\cdot)$ can be written as

$$R = ASA^* + N \tag{4}$$

where S and N are the covariance matrices of the sources and the noise, respectively. The noise is usually assumed spatially and temporally uncorrelated with intensity $N = \sigma I$. It is obvious that if $D < M$ and S is positive definite, then the matrix $R - \sigma I$ will have rank D, and therefore it has a nullspace of dimension $M - D$. It also readily follows that all columns of A are orthogonal to this nullspace. As noted by Schmidt [10] and Bienvenu and Kopp [11] (see also [12] and [13]), the above observations lead to the following way to determine, from a perfectly known R, the number of sources D, their directions $\{\theta_i\}$, the source covariance S, and the noise power σ.

1) Compute the eigenvalues and eigenvectors of the $M \times M$ matrix R.

2) D and σ are determined by the facts that the minimum eigenvalue, equal to σ, has multiplicity $M - D$.

3) The $M - D$ eigenvectors corresponding to the minimal eigenvalues are orthogonal to all the D signature vectors $A(\theta_i)$. Therefore, by a linear search on the array manifold, we can determine the D directions orthogonal to the subspace determined by the eigenvectors corresponding to the $M - D$ minimal eigenvalues.

4) Finally, we can compute

$$S = (A^*A)^{-1} A^*(R - \sigma I)\, A(AA^*)^{-1}. \tag{5}$$

The determination of the source directions θ_i is usually performed by plotting, as a function of θ, some measure of orthogonality of $A(\theta)$ to the subspace determined by the eigenvectors E_i. This measure is often chosen to be

$$\Phi(\theta) = \frac{A(\theta)^*\, A(\theta)}{\sum_{j=1}^{M-D} (A(\theta)^* E_j)^2}. \tag{6}$$

The K points at which $\Phi(\theta)$ peaks to infinity (ideally) are the directions of the sources.

In the real-world situation, the steps of the idealized procedure presented above have to be replaced by corresponding estimation methods. References [9]–[16] provide thorough discussions of these, so we only briefly note two key steps.

1) Estimate R, say, by the sample covariance matrix $(1/K)\, \Sigma_{i=1}^{K}\, r(t_i)\, r^*(t_i)$.

2) Either perform a hypothesis testing, based on likelihood ratios with thresholding, to achieve desired significance levels [10] or use a model-order identification method [14], based on some information-theoretic criteria (Akaike's I.C. or Rissanen's minimum description length) to determine the multiplicity of the smallest eigenvalue (i.e., the number of elements in the cluster of smallest eigenvalues of $\hat{R}$). This provides an estimate of D, and the arithmetic mean of the $M - D$ smallest eigenvalues is an estimate of the noise power σ.

Application to the Overlapping Echos Problem

Here we have available, by assumption, a sequence of outputs of a single sensor. Over successive data windows of length T, which are synchronized to the repetitive cycle of radar transmission and reception, we sample the received waveforms $r_j(\cdot)$ at M instants and stack the successive samples in a vector of length M. Then, we have

$$r_j = \begin{bmatrix} r_j(t_1) \\ r_j(t_2) \\ \vdots \\ r_j(t_M) \end{bmatrix} = [A(\theta_1)\, A(\theta_2) \cdots A(\theta_D)] \begin{bmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{Dj} \end{bmatrix} + \begin{bmatrix} n_j(t_1) \\ n_j(t_2) \\ \vdots \\ n_j(t_M) \end{bmatrix} \tag{7}$$

where the time-domain "signature" of an echo delayed by θ is simply the vector

$$A(\theta) = [s(t_1 - \theta), s(t_2 - \theta) \cdots s(t_M - \theta)]^* \tag{8}$$

where * denotes matrix transpose.

The above representation is a direct consequence of the assumed model for the signal equation (1). In this formulation, it becomes clear that the sampled waveform's covariance eigenstructure contains all the information needed for the recovery of the delay parameters, the noise level, and the second-order statistics of the random gains m_{ij}, provided only that $M > D$.

Therefore, the one-sensor multiexperiment signal resolution problem is seen to have an identical structure to the problem of multitarget direction finding with an array of antennas. The nice feature of this correspondence is the fact that the "array manifold" for the signal resolution problem is trivially obtained from the signal shape, which is assumed to be given. Furthermore, the desired array-manifold property—linear independence of the $A(\theta)$'s for any set of less than M delays—is readily satisfied, provided we work with any finite-span pulse-type waveforms.

In practical radar systems, the probing signal is an amplitude modulated high frequency sinusoidal signal, with the envelope a pulse of shape $s(\cdot)$, i.e.,

$$s_{RF}(t) = s(t) \cos(\omega_0 t + \phi). \tag{9}$$

In this case, the echos received at the radar antenna are modeled by the following equation:

$$r_{RF}(t) = \sum_{i=1}^{D} m_i s(t - \theta_i) \cos(\omega_0(t - \theta_i) + \phi + \psi_i) + n(t) \tag{10}$$

where the ψ_i's stand for random phases due to propagation through the medium. Now, the signal may be either sampled in its original version or a conversion to baseband may be performed. In both of these cases, the random phase will have the effect of multiplying the signal envelope by an additional random factor. It is also quite natural to assume that the random phases corresponding to different echos are uncorrelated. This shows that for the RF case—even if the random factors due to, say, Rayleigh fading, would be slowly changing and this would make their covariance matrix nearly singular—the gains m_i due to random phases will provide a nonsingular overall gain covariance, rendering the MUSIC algorithm applicable.

To determine the number of echos and estimate their location, we apply the MUSIC algorithm, as described above. First, estimate the received signal covariance by $1/K\, \Sigma_{j=1}^{K}\, r_j r_j^*$, and perform a spectral decomposition on the estimated covariance. This yields

$$\hat{R} = \sum_{i=1}^{M} \lambda_i V_i V_i^*$$

where we order the eigenvalues as $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_M$.

Determination of Number of Echos and Noise Level

The number of echos present in the returning waveforms can be estimated by using one of the two approaches described below.

1) Perform a sequence of hypothesis tests by comparing the likelihood ratio for the first p eigenvalues being equal for $p = M, M - 1, \cdots, 1$, to a suitably chosen set of thresholds. The likelihood ratio is, under Gaussian assumptions for all random factors, given by

$$LR(p) = \left\{ \frac{\left[\prod_1^p \lambda_i \right]^{1/p}}{(1/p) \sum_1^p \lambda_i} \right\}^{p\frac{K}{2}}. \tag{11}$$

The choice of thresholds in this so-called Bartlett–Lawley method is rather complicated and is dictated by the desired significance level from a χ^2 distribution (see [10] and [14]), and the first p for which $LR(p)$ exceeds the threshold is accepted as an estimate of the multiplicity of the minimal eigenvalue.

2) Compute for $p = M, M - 1, \cdots, 1$ the value of an information-theoretic criterion such as Akaikes's IC or Rissanen's MDL, given, respectively, by

$$\text{AIC}(p) = \log \{LR(p)\} + \tfrac{1}{2}(M - p)(M + p + 1) \tag{12}$$

$$\text{MDL}(p) = \log \{LR(p)\} + \tfrac{1}{4}(M - p)(M + p + 1) \log K \tag{13}$$

and estimate the multiplicity of the minimal eigenvalue by the value of p that minimizes the chosen (preferred) criterion. This approach, paralleling model order identification methods in time series analysis, was recently proposed by Wax and Kailath; see [14].

Once an estimate for the multiplicity of the minimal eigenvalue becomes available, the number of sources is estimated by $\hat{D} = M - p$, and the maximum likelihood estimate of the noise power is simply the mean of the p smallest eigenvalues [9]-[13]:

$$\hat{\sigma} = \frac{1}{p} \sum_{i=1}^{p} \lambda_i. \tag{14}$$

We note that, although the above procedures are derived under Gaussian assumptions, they are, in fact, reasonable tests for more complicated situations too. The likelihood ratio, for example, by comparing the geometric mean of p smallest eigenvalues to their arithmetic mean, provides a measure of their uniformity. The closer this measure is to 1, the more similar are $\lambda_1, \lambda_2 \cdots \lambda_p$.

Estimation of Echo Delays

From a singular value decomposition, the estimate of the nullspace of $\hat{R}$ is obtained as the span of $\{\hat{E}_i\}_{i=1,2,\cdots M-\hat{D}}$ with $\hat{E}_i = V_i$. The estimation of echo delays is performed by a search over $\theta \in [0, T]$ for the $\hat{D}$ peaks of the following measure of orthogonality to the nullspace:

$$\Phi(\theta) = \frac{A(\theta)^* A(\theta)}{\sum_{j=1}^{M-\hat{D}} (A(\theta)^* \hat{E}_j)^2}. \tag{15}$$

It is important to note that the above algorithm implies a search over all possible values of the delays, i.e., $\theta \in [0, T]$, and that, in practice, we will have to search on a sampled parameter space. The sampling of the parameter space is, however, *independent* of the sampling done for the received signal. Provided the sampling of the signals and that of the parameter space jointly provide an "array manifold" with the required linear independence properties, the algorithm will produce high resolution estimates of the echo delays. (For signals that are not piecewise constant, this is always the case.)

In the above development, we implicitly assumed that the number of echos present is less than the number of samples we take from the received signal, a requirement that can be easily met by sampling the data at a sufficiently high rate.

We note that the search procedure involves the formation of $M - D$ inner products for each point in the parameter space, and if we want high resolution, this will imply a rather lengthy computation. Research is needed on more efficient search techniques and perhaps on special-purpose hardware for such purposes.

An estimate of the covariance structure of the random gains $\{m_i\}$ also becomes available in the process of applying the MUSIC algorithm [via (5)], and it can be used to obtain more information on the nature of the scatterers. For example, in a radar problem, some of the echos are direct target images; others represent multipath propagation of the signals reflected by those targets. It is expected that the gains of the multipath echos are strongly correlated with that of the directly returned signal, so that the gain correlation matrix estimate can help determine the number of real targets in a multipath environment. In geophysical applications, the echos from a layered medium comprise both direct returns and reverberations; therefore, seismic sounding experiments may be regarded as results of remote sensing in a dense multipath environment. In this case too, an estimate of the gain statistics is extremely useful for recovering the properties of the medium.

III. Some Variations of the Signal Resolution Problem

The procedure presented in the previous section can be applied in a variety of situations in which some further information is available on the way the obstacles return and modify the probing signal and the effect of free propagation of the signal in the medium. Also, the problem of determining the number and identity of waveforms coming from a finite collection of signals is seen to be readily accommodated within the eigenstructure approach. The search in this case will simply be conducted on the set of possible signals which will constitute the underlying "signal manifold."

Dispersive Propagation and Obstacle Signatures

Suppose that the propagation in the medium makes predictable changes in the signal shape. This information may become available either through measurements or by proper modeling of the propagation in a dispersive environment. Also assume that there are P types of obstacles whose effect, or signature, on the returned signal is described by a convolution of the impinging signal with a given kernel parametrized by the type of obstacle. Then, if the probing signal is $s(\cdot)$, the signal received at a delay θ and due to a single obstacle of type L will have the form

$$s_{\theta,L}(t) = m\Phi_{\theta/2}\{H_L(t) * [\Phi_{\theta/2}\{s(t)\}]\} \tag{16}$$

where Φ_θ is an operator describing the modifications in signal shape due to propagation in the medium for a time θ, $H_L(\cdot)$ describes the modification due to an obstacle of type $L \in \{1, 2, \cdots, P\}$, and, again, m is a random gain factor.

With these assumptions, the resolution of a series of overlapping echos and the determination of the types of the respective obstacles can be done invoking the MUSIC algorithm. Equation (16) simply redefines the "signal manifold" over which the MUSIC search will have to be conducted. In the above case, it will imply P linear searches, one for each type of obstacle. The simplest case of dispersive propagation might be described by

$$\Phi_\theta\{x(t)\} = \frac{1}{q(\theta)} x\left(\frac{t-\theta}{q(\theta)}\right) \tag{17}$$

which implies that the signal simply "spreads out" according to a given function of the propagation time $q(\cdot)$. The problem of obstacle signatures is, of course, of great importance in radar target identification. To apply the MUSIC algorithm to this problem, one needs good characterizations of target influences on the returned signals, i.e., of the kernels $H_L(\cdot)$. These may be obtained from calibration experiments or via some theoretical considerations; however, it is fair to note that in many practical situations such information may be very difficult to gather.

Resolution of Uncertain Signals

Consider the following problem: we are given a set of signals $\{s(t, \theta)\}$, with the parameter θ ranging over $\{1, 2, \cdots, P\}$, that are known to be approximations of a set of signals $\{s_0(t, \theta)\}$. Suppose further that for each value of the parameter, this approximation was chosen as follows:

$$s(t, \theta) = s_0(t, \theta) + u_\theta(t) \tag{18}$$

where $u_\theta(\cdot)$ is a sample of white noise with power Π_θ. The $u_\theta(\cdot)$ sequences are thought of as a model for the uncertainty in our knowledge of $s_0(t, \theta)$. Assume that we are given a sample of K observations

$$\{r_j(t)\}_{j=1,2\cdots K} = \left\{\sum_{i=1}^{D} m_{ij}\, s(t, \theta_i) + n_j(t)\right\}_{j=1,2\cdots K}$$

$$\theta_i \in \{1, 2, \cdots, P\} \tag{19}$$

and we wish to obtain estimates of the number of signals present and their identity. Note that we have uncertain versions of the signals that compose the observations. For the estimate of the number of signals present, this is irrelevant; however, the search procedure should clearly be modified to account for the degrees of uncertainty to which we know the signals. Such a problem also occurs in the framework of eigenstructure methods for direction finding with noisy array calibrations. The result of a simple analysis is that the search has to be done using a weighted cost function Π_θ:

$$\Phi(\theta) = \frac{1}{\Pi_\theta} \frac{A(\theta)^* A(\theta)}{\sum_{j=1}^{M-D} (A(\theta)^* \hat{E}_j)^2} \tag{20}$$

where $A(\theta) = [s(t_1, \theta)\, s(t_2, \theta) \cdots s(t_M, \theta)]^*$. This is not an unexpected result since it shows that the more uncertain we are about a signal, the more reluctant we shall be in giving it as an estimate for one of the signals present. This solution can also be adapted to the echo delay estimation for the case in which we assume that the returning signal gets more and more distorted, in an unpredictable way, as its return delay increases.

A variety of further problems may be addressed within the framework of eigenstructure methods; here are just a few of them.

1) Location of displaced and scaled overlapping objects of known shapes in a scanned image.

2) Resolving for the elements of a composite probability density function from observations of random realizations.

3) Tracking of multiple target echos by repeated application of the MUSIC algorithm, with reduced search requirements, based on prior information on delays.

4) Estimation of abrupt impedance changes for layered earth models or transmission lines from repeated scattering experiments.

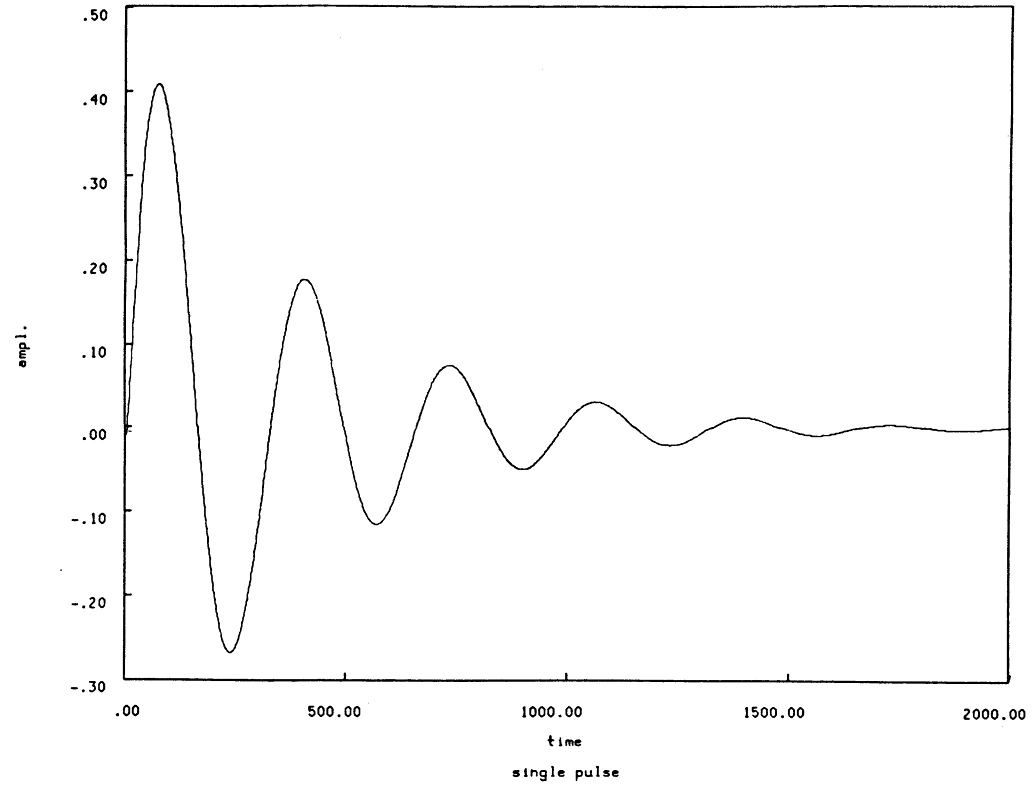
IV. Simulation Results

To demonstrate the performance of the proposed signal resolution method, we performed a set of simulations on artificially generated data sets. Three types of probing pulse shapes were chosen: a decaying sine wave, a triangular pulse, and a bell-shaped (Gaussian) pulse, all of significant time span of 1000 ms.

For the first set of simulations, the data were generated by superimposing three pulses of the same shape, delayed by arbitrarily chosen times, in the span [0, 1000], and weighted by random gains with Rayleigh distribution having a mean of 1.0. (This type of random weighting is commonly used to model fading due to propagation of signals in random lossy media.) To the resulting signals over the interval [0, 2000], white noise was added, with various intensities resulting in signal-to-noise ratios (SNR) that varied between 0-40 dB. (The definition of SNR adopted here was the ratio of average signal energy and the noise variance over the sampling window.) For each SNR, the MUSIC algorithm was applied to the estimated covariance when this estimate was obtained from 50, 100, 200, 500, and 1000 independently generated data sets (snapshots). The sampling interval was uniform at 80 ms, with a total of $M = 25$ samples gathered per data set. First, the number of signals was estimated using the AIC and the MDL criteria; then a search on the delay space of [0, 1000] was performed. The sampling of the parameter space was done at intervals of 1.0 ms (as we stressed before, the method enables us to resolve signals delayed by fractions of the data-sampling interval). Plots of the distance measure given by (15) were obtained, and we superimposed on the same coordinates the results obtained with the various numbers of data sets for the same SNR.

Fig. 1 shows the results of applying the signal resolution algorithm to the case of superimposed decaying sinusoidal signals. The delays were $\theta_1 = 200$, $\theta_2 = 260$, and $\theta_3 = 400$. Fig. 1(a) shows the probing signal shape $s(\cdot)$, and Fig. 1(b) shows the values of the information-theoretic criteria (AIC and MDL) for determining the number of returns present, for 10 dB SNR, for the most difficult case of only 50 snapshots. Both criteria were found to provide perfect estimates for the number of signals present in all the scenarios that were tested. Fig. 1(c) and (d) shows typical data waveforms at 0 and 10 dB SNR, with three overlapping delayed and weighted pulses. Fig. 1(e) and (f) presents the results of the search on the parameter space using (15).

The second simulation involved three types of probing pulse shapes: a decaying sine wave, a triangular pulse,



(a)

Fig. 1. (a) The decaying sine test signal $s_{SIN}(t)$. (b) AIC and MDL for SNR = 10 dB and 50 snapshots. (c), (d) Typical data waveforms at 0 and 10 dB SNR. (e), (f) Results of the MUSIC search for 0 and 10 dB SNR for K = 50, 100, 200, 500, 1000. The delays were 200, 260, and 400 ms.

and a bell-shaped (Gaussian) pulse, all of significant time span of 1000 ms. There were nine signals, three of each type, overlapping at various delays. In this case, the MUSIC algorithm performs three one-dimensional searches for their resolution. The results are plotted in Fig. 2(a)–(c) for various ensemble sizes (K = 50, 100, 200) at an SNR of 20 dB.

From the simulations that we performed, it appears that the method has the following tradeoffs: at low SNR's, a high number of snapshots are required for good performance, whereas at high SNR's, about 100 returns already provide good resolution and delay estimates. Also, it is clear that the method needs either more data or higher SNR to resolve very closely spaced signals. There is no problem, however, in resolving delays that are noninteger multiples of the sampling intervals, and this is clearly an advantage of the eigenstructure method.

We attempted to make some comparisons to conventional matched filter techniques, but due to the random gains of the signals, these techniques failed very badly.

V. Conclusions

In this paper, we presented a novel solution to the signal resolution problem which arises in many radar and sonar signal processing, geophysics, and imaging problems. The algorithm is a time-domain version of a procedure developed for direction finding with a passive antenna array [10], [11], and is, in fact, a particular case of a statistical factor analysis methods [15], [16]. To apply this method, we had to redefine the concept of "array manifold" to be the set of possible deterministic signals (or factors), and to assume that the data are a randomly weighted combination of an arbitrary number of those signals measured in additive noise with known covariance structure. This is indeed the natural form in which data are obtained in many practical applications.

We should mention that notions of finite dimensional signals and their orthogonal spaces were also exploited in the context of identifying exponential components in a waveform for system identification and spectral analysis applications. Many useful results along these lines were obtained in parallel and independently of the work done by the array processing community, as exemplified by [9], [17]–[19]. However, it should be noted that model-based methods, MUSIC or Kalman filtering, can be quite sensitive to differences between the actual and assumed models, and more work is needed to obtain robust solutions; see, e.g., [20].

Some further problems to which the MUSIC algorithm can be applied were briefly discussed. An important application of these methods would be their incorporation into multitarget tracking procedures. Suppose we can assume a Markovian (state-space) model for the random movement of the targets in space. Then, considering the

For K = 50 and S/N = 10 *dB*:

AIC(0)	AIC(1)	AIC(2)	AIC(3)	AIC(4)	AIC(5)	AIC(6)	AIC(7)
907.43	516.98	358.50	143.01	159.29	175.30	189.47	203.22
MDL(0)	MDL(1)	MDL(2)	MDL(3)	MDL(4)	MDL(5)	MDL(6)	MDL(7)
907.43	504.49	334.00	107.01	112.29	117.80	121.97	126.22

AIC(8)	AIC(9)	AIC(10)	AIC(11)	AIC(12)	AIC(13)	AIC(14)	AIC(15)
215.40	227.31	238.79	249.90	260.43	269.49	278.76	287.57
MDL(8)	MDL(9)	MDL(10)	MDL(11)	MDL(12)	MDL(13)	MDL(14)	MDL(15)
129.40	132.81	136.29	139.90	143.43	145.99	149.26	152.57

AIC(16)	AIC(17)	AIC(18)	AIC(19)	AIC(20)	AIC(21)	AIC(22)	AIC(23)	AIC(24)
294.76	300.63	304.23	309.93	314.06	316.85	319.78	322.18	324.00
MDL(16)	MDL(17)	MDL(18)	MDL(19)	MDL(20)	MDL(21)	MDL(22)	MDL(23)	MDL(24)
154.76	156.13	155.73	157.93	159.06	159.35	160.28	161.18	162.00

(b)

Fig. 1. (*Continued.*)

slowly varying covariance of the observed data as a nonlinear function of the state (target positions), one can develop an extended Kalman filter for target tracking which uses MUSIC as an intermediate step.

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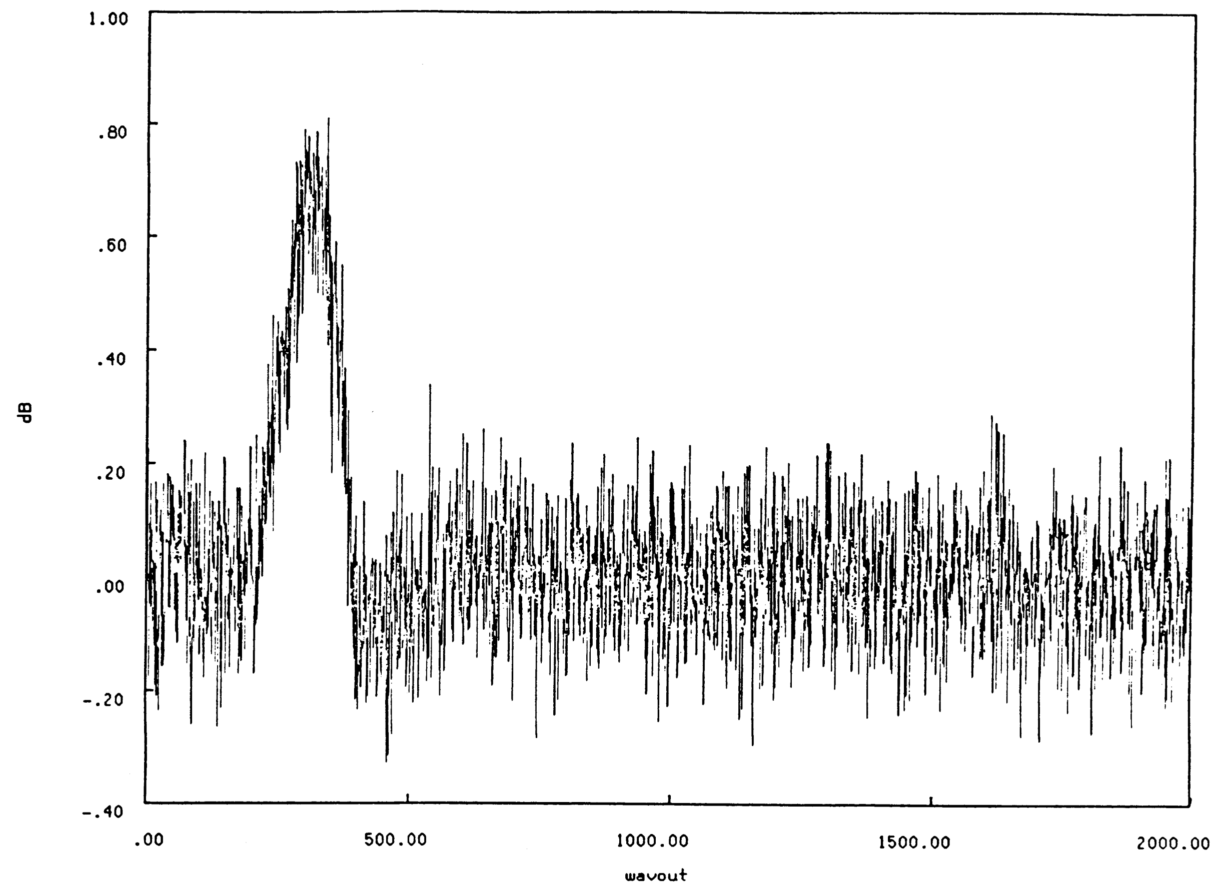
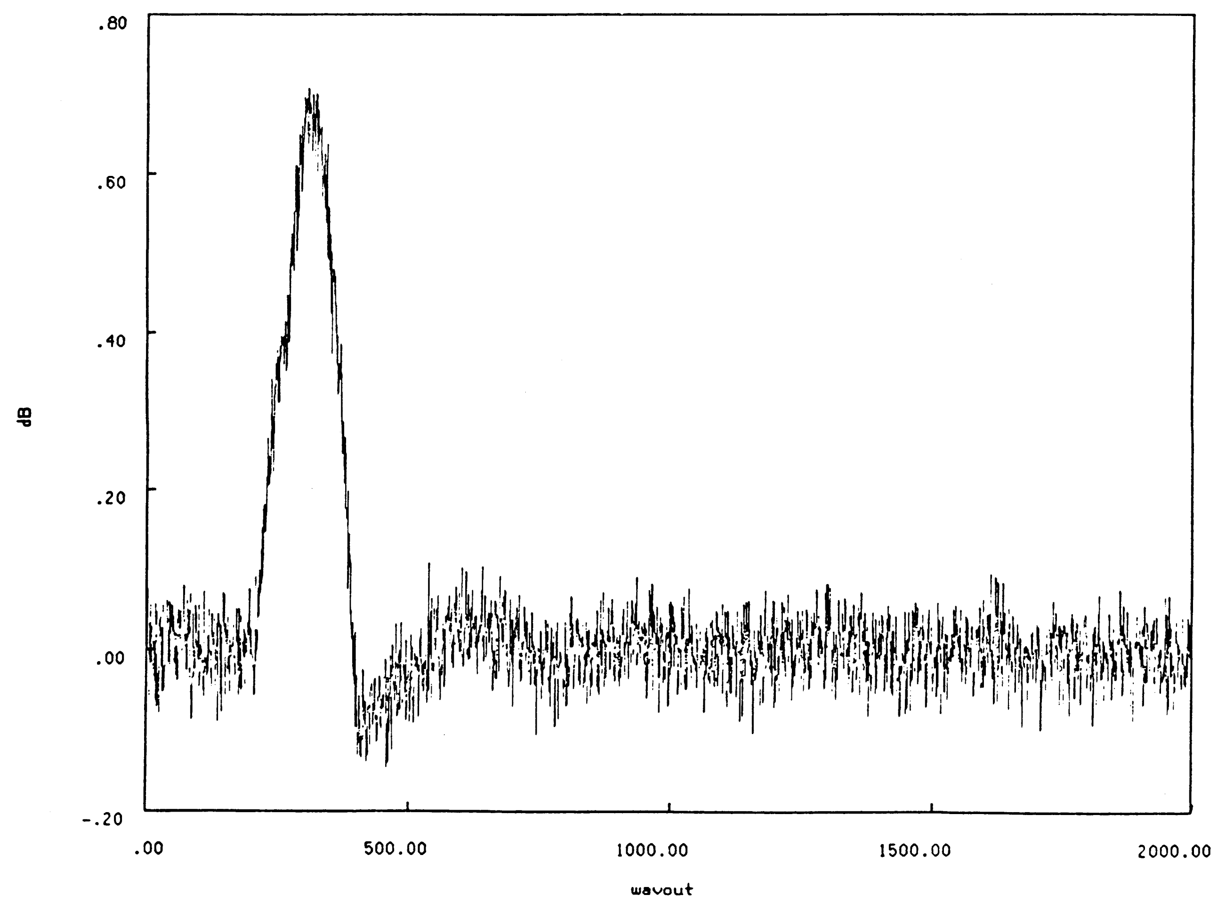
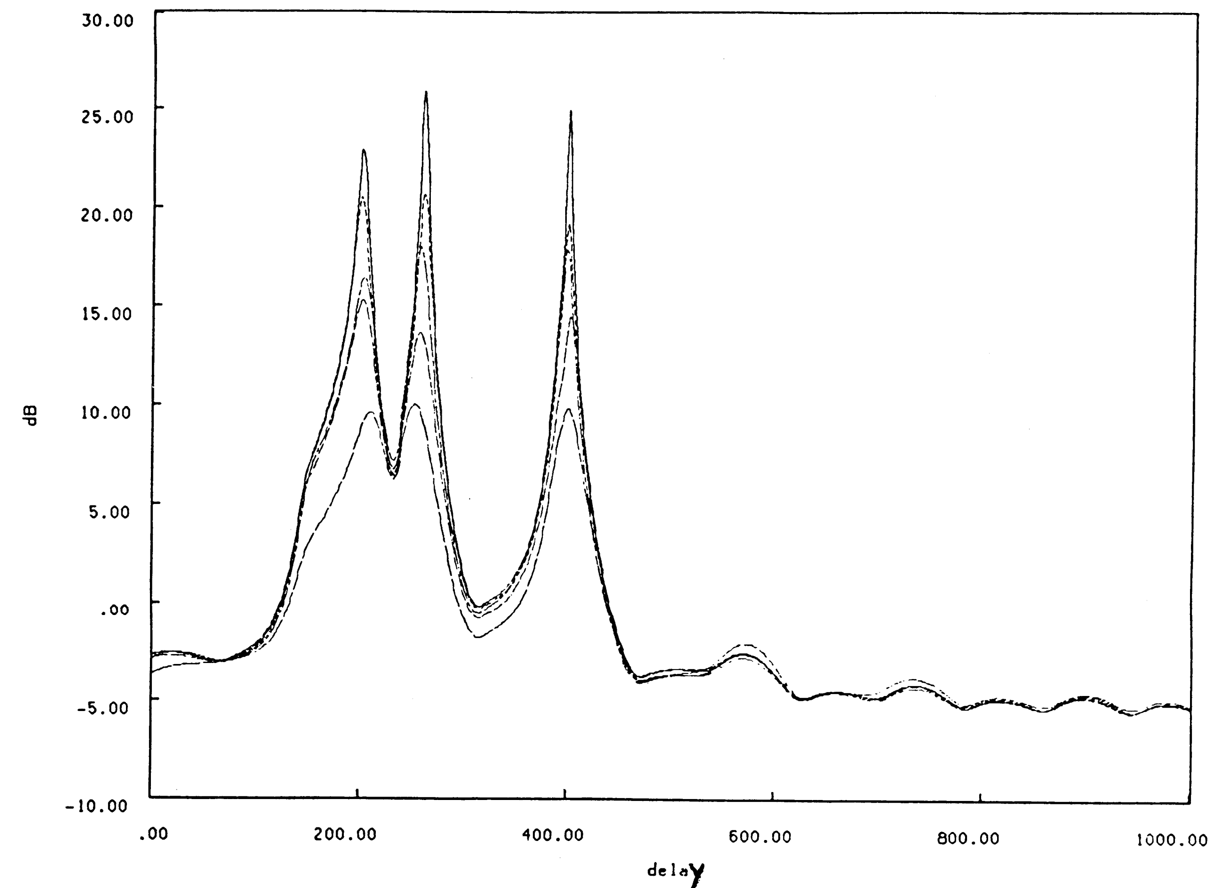


(c)



(d)

Fig. 1. (*Continued.*)



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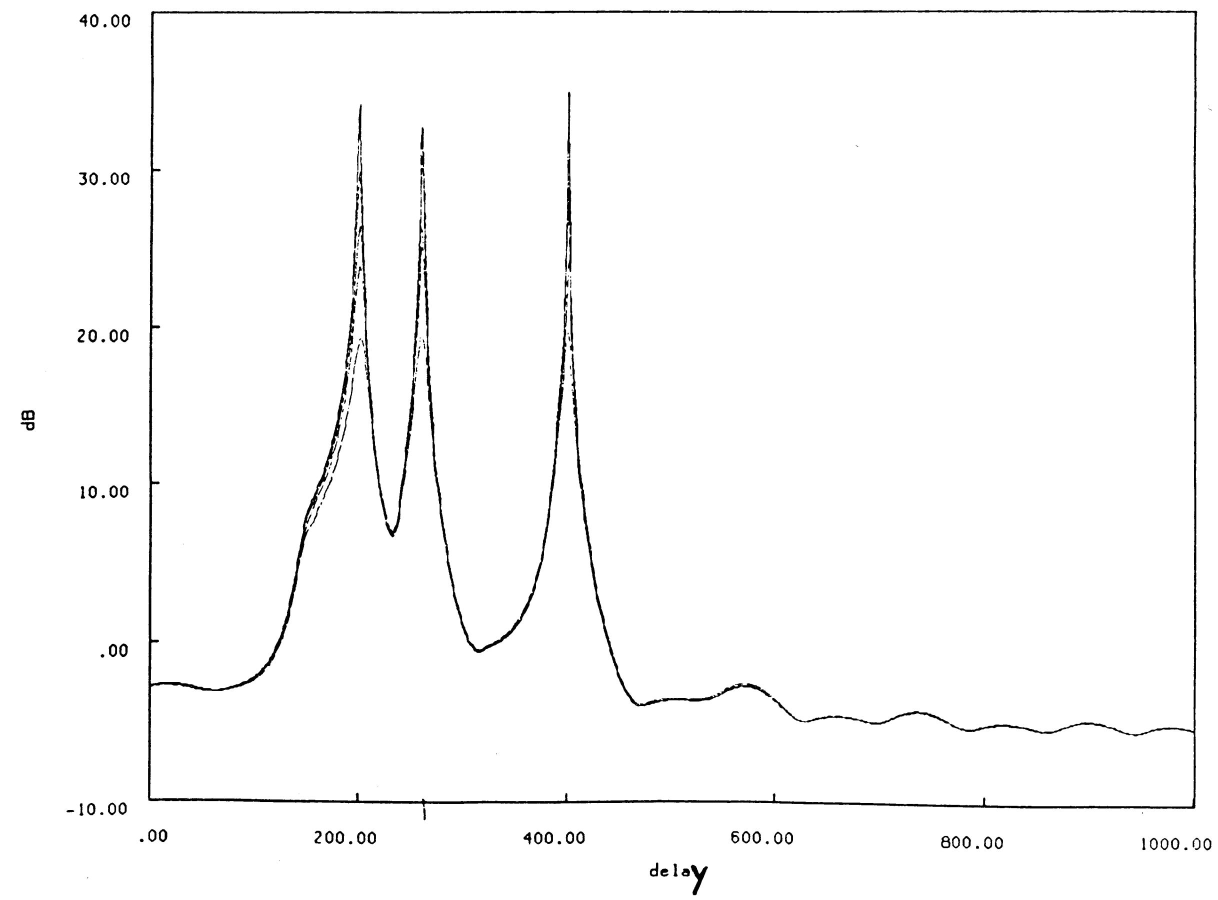

(f)

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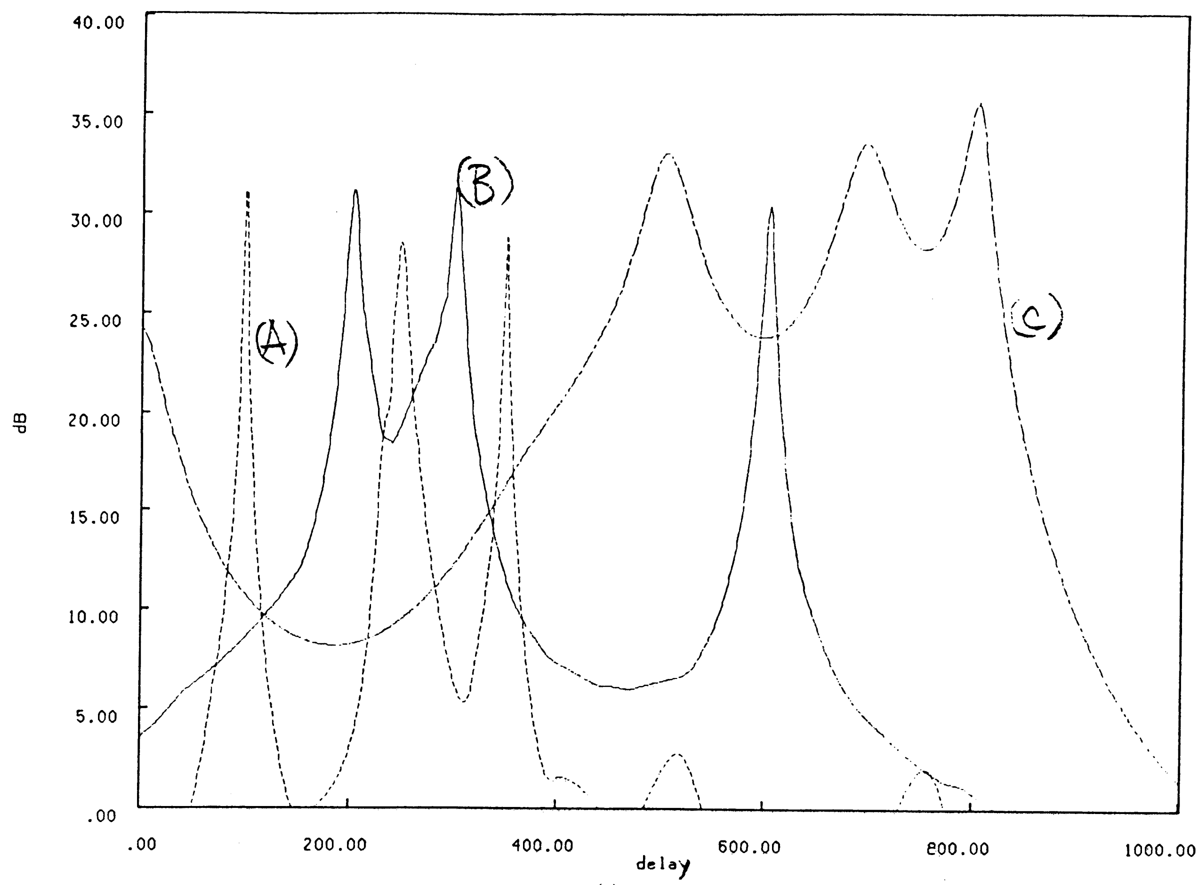
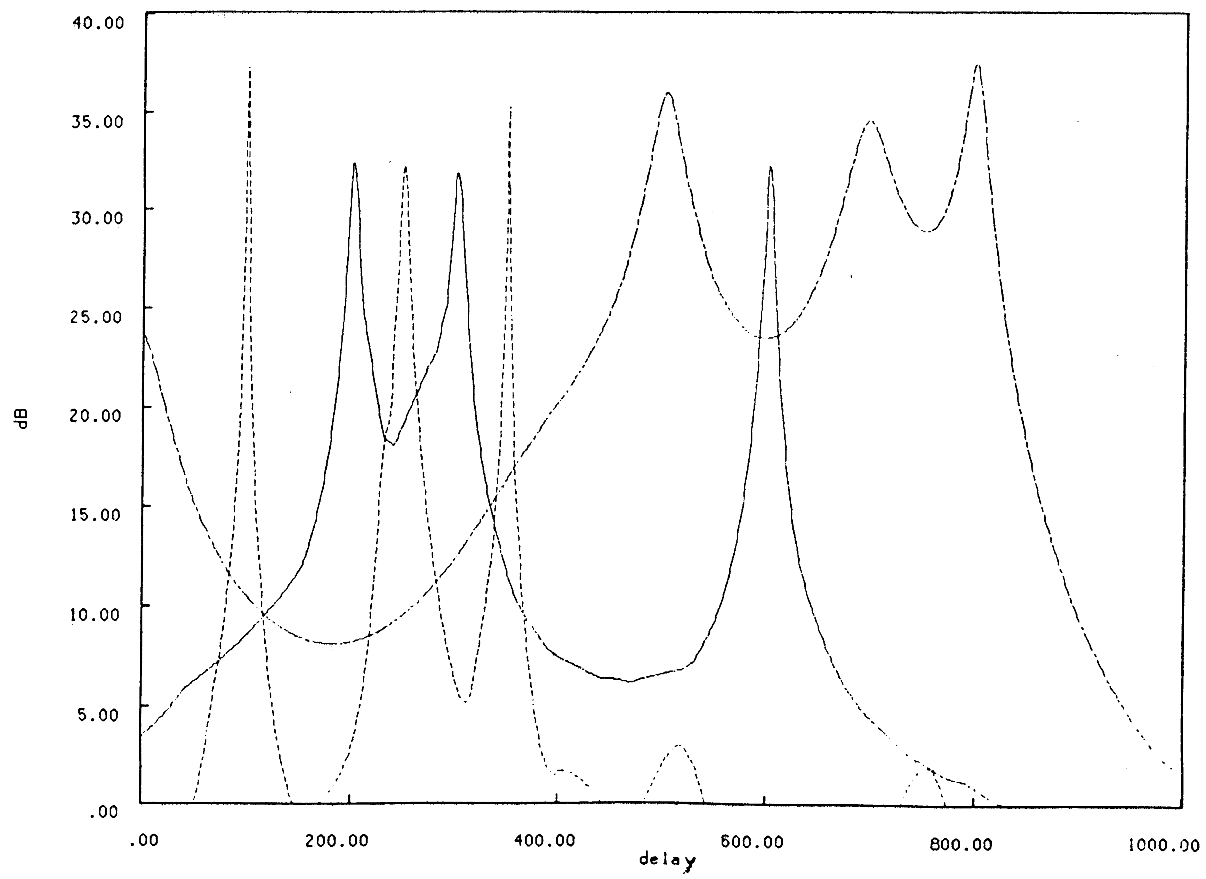
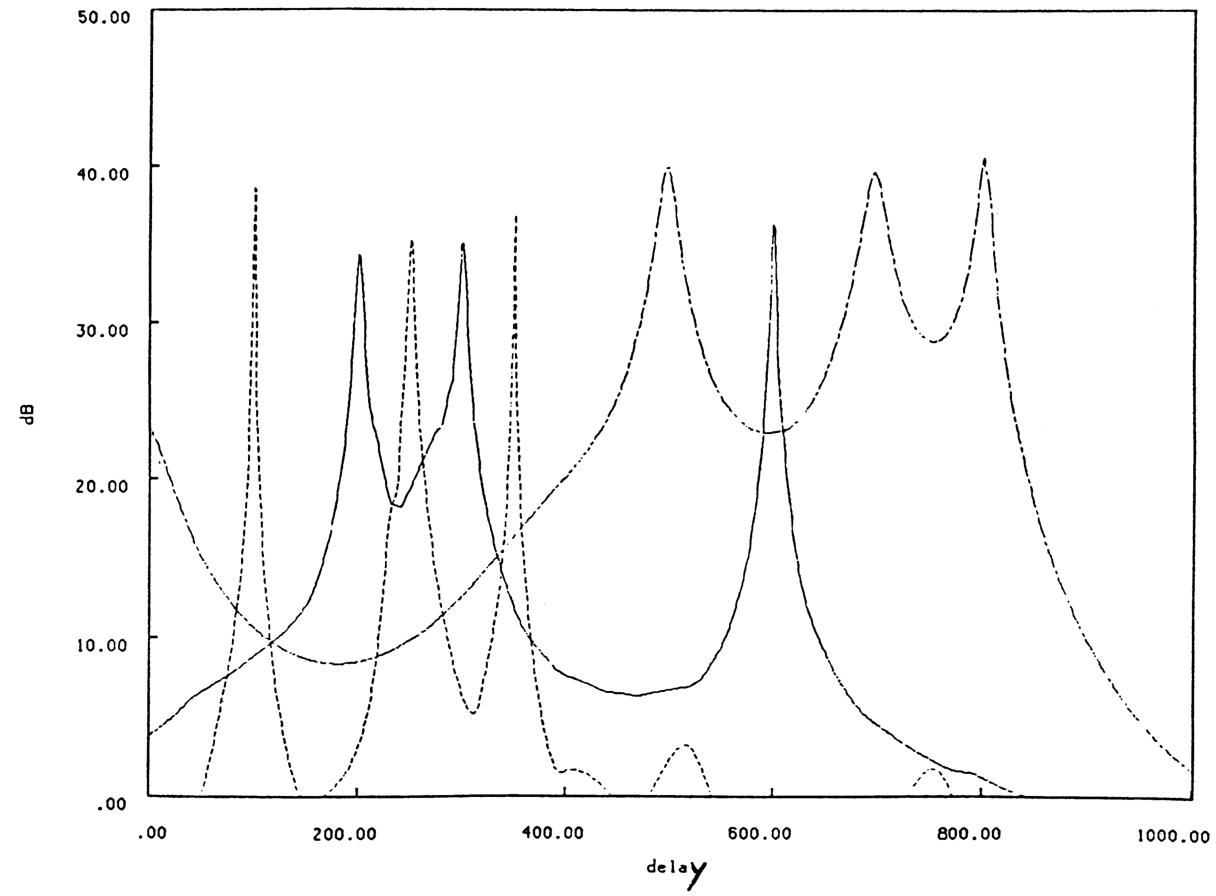


(a)



(b)

Fig. 2. (a), (b), (c) Results of MUSIC search for three types of pulses at various delays. SNR = 20, K = 50, 100, 200. The three curves are the results of three independent searches over "signal manifolds" corresponding to the decaying sine (A), triangular (B), and Gaussian pulse shapes (C).



(c)

Fig. 2. (*Continued.*)

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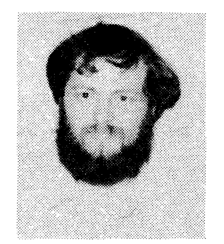

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Tie-Jun Shan, for a photograph and biography, see p. 536 of the June 1985 issue of this TRANSACTIONS.

Thomas Kailath (S'57–M'62–F'70), for a photograph and biography, see p. 536 of the June 1985 issue of this TRANSACTIONS.