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Stabilizing state-feedback design via the moving horizon method†

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A stabilizing control design for general linear time varying systems is presented and analysed. The control is a state-feedback law with gains determined by a standard method employed in optimal regulator problems. The considered cost function is, however, dynamically redefined over a fixed depth horizon. The method is shown to yield a stable closed-loop system and computationally efficient recursions for the feedback gain are provided.

1. Introduction

Several approaches exist for the design of stabilizing control laws for linear time-invariant systems. Along with the classical frequency-domain techniques, the ' modern ' performance index optimization methods further guaranteed robust, stabilizing feedback controls (Kwakernaak and Sivan 1972). The situation is quite different, however, for time-varying linear systems. While an ' optimal ', stabilizing state-feedback gain for a time-invariant system can be found by solving an algebraic Riccati equation, the corresponding solution for the time-varying case requires the backwards iteration of a matrix differential equation over an infinite horizon (Kalman 1960, Athans 1971). Obviously, this is not a very practical way to obtain a stabilizing control. The problem arises from the fact that the computation of the gain at every instant of time requires, in principle, the optimization of a performance index over an infinite time span into the future. A natural way to try to overcome the computational difficulty is to assume that at all moments t we have to find the optimal control for a fixed, finite horizon of depth T (Kwon and Pearson 1977, 1978).

The standard regulator problem poses the question of determining the optimal control $u^\circ(\cdot)$ to be applied to a linear system in order to minimize a cost functional over a given interval (t_i, t_f). In the case of a quadratic cost, the resulting control is a simple state-feedback law, the gain computation involving the well known backwards Riccati equation (Kwakernaak and Sivan 1972). The control applied at time t, given a ' sliding ' horizon of fixed depth T, would therefore be the initial step in minimizing a quadratic performance index over $(t, t+T)$. It is also immediately apparent that this procedure leads to a state-feedback control law, the gain being computed through a Riccati recursion starting at $t+T$ and proceeding backwards to t (Kwon and Pearson 1977).

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Although very reasonable in concept, this receding horizon procedure does not have any obvious interpretation in terms of optimizing a performance index over some predetermined interval (t_i, t_f) ; its value lies instead in the fact that it yields a practical and computationally efficient technique for stabilizing general time-varying linear systems. Also, it is worth mentioning that this is a proper generalization of Kleinman's ' easy ' way to stabilize a time-invariant systems through state-feedback (Kleinman 1970).

Since computing the feedback gain for this defined control law requires, in principle, completing a backwards recursion over an interval of length T for all t, it is not obvious that this procedure can be rendered computationally feasible. In the sequel we shall show that in fact one can derive a recursive algorithm that updates the control gain directly, avoiding the necessity of solving a backwards Riccati equation over and over again. This algorithm is derived through a convenient embedding of the feedback gain in a suitably defined scattering matrix and using some well known results of Redheffer scattering theory (Redheffer 1961). We shall then address the problem of system stability, establishing that, under certain uniform controllability conditions and some conditions on the moving interval cost function, the closed-loop system becomes asymptotically stable.

A suboptimal state estimator, the structural dual system of the receding horizon controller, is introduced and briefly discussed in the last section of the paper.

2. Moving horizon control laws

Consider the time-varying linear system described by

$$\frac{d}{dt} x(t) = A_t x(t) + B_t u(t) \tag{2.1}$$

where $x(\cdot) \in R^n$ and $u(\cdot) \in R^m$. Let J be a standard quadratic cost functional over a fixed interval (t_i, t_f) defined as

$$J = \int_{t_i}^{t_f} [x'(\tau) Q_\tau x(\tau) + u'(\tau) R_\tau u(\tau)]\, d\tau + x'(t_f) F_{t_f} x(t_f) \tag{2.2}$$

Here Q_t, R_t and F_t are known, positive-definite, symmetric matrices, essentially design parameters.

It is well known (Kwakernaak and Sivan 1972) that the optimal control input that minimizes J is provided by the following state-feedback law

$$u^\circ(t) = -R_t^{-1} B'_t K(t, t_f ;\ F_{t_f}) x(t) \tag{2.3}$$

The gain $K(t, t_f ;\ F_{t_f})$ is computed through the backwards Riccati recursion

$$\begin{aligned} -\frac{d}{d\tau} K(\tau, t_f ;\ F_{t_f}) = {} & K(\tau, t_f ;\ F_{t_f}) A_\tau + A'_\tau K(\tau, t_f ;\ F_{t_f}) \\ & - K(\tau, t_f ;\ F_{t_f}) B_\tau R_\tau^{-1} B'_\tau K(\tau, t_f ;\ F_{t_f}) + Q_\tau \end{aligned} \tag{2.4}$$

with final condition $K(t_f, t_f ;\ F_{t_f}) = F_{t_f}$.

The modified, receding horizon control $u^m(t)$ is defined as the input at time t that would be needed to minimize, over $(t, t+T)$, the following criterion

$$J_m = \int_t^{t+T} [x'(\tau)Q_\tau x(\tau) + u'(\tau)R_\tau u(\tau)]\, d\tau + x'(t+T)F_{t+T}x(t+T) \qquad (2.5)$$

Therefore, this control law is also a state-feedback law, given by

$$u^m(t) = -R_t^{-1} B'_t K(t, t+T\,;\, F_{t+T})x(t) \qquad (2.6)$$

the gain $K(t, t+T\,;\; F_{t+T})$ being obtained from the backwards Riccati equation (2.4) with t_f replaced by $t+T$.

The interpretation of this modified control is that, at each moment t, the input applied is chosen as if the optimization of the criterion J_m over $(t, t+T)$ was the overall objective. It is readily seen, though, that this control does not minimize any obvious overall cost function of the type (2.2) on any given interval (t_i, t_f).

In principle, in order to compute $K(t, t+T\,;\; F_{t+T})$ one might solve at all moments in time the backwards Riccati equation, with final condition at $t+T$ given by F_{t+T}. This is not a computationally feasible approach. However, one immediately realizes that in the case of time-invariant systems and fixed weighting matrices Q, R and F, the moving horizon method yields a *constant* feedback gain. The control law, in this case, is simply given by

$$u^m(t) = -R^{-1}B'K_T x(t) \qquad (2.7)$$

where K_T can be obtained from

$$\frac{d}{dt} K_t = K_t A + A'K_t - K_t BR^{-1}BK_t + Q\,; \quad K_0 = F \qquad (2.8)$$

The particular case of $Q=0$ and $F\to\infty$ provides Kleinman's method for stabilization of a time-invariant system (Kleinman 1970).

The general case with $F_t = \infty$ turns out to be very important in providing a stabilizing moving horizon control for time-varying systems. The infinite weight assigned to the final state implies that the optimal control is required to take the state to zero at the final time, while minimizing a quadratic cost over the given interval (Kwon and Pearson 1977). In this case, the backwards Riccati equation has infinity as its initial condition, a somewhat ambiguous starting point for a recursion. Therefore, the following well known manipulation is invoked: instead of considering the differential equation for $K(t, t_f\,;\; \infty)$, derive a recursion for $P(t, t_f) = K^{-1}(t, t_f\,;\; \infty)$. It is quite simple to show that the result is the following Riccati equation

$$-\frac{d}{dt} P(t, t_f) = -P(t, t_f)A'_t - A_t P(t, t_f) - P(t, t_f)Q_t P(t, t_f) + B_t R_t^{-1} B_t \qquad (2.9)$$

Now however, the final condition is $P(t_f, t_f) = 0$. For the case of $F_{t_f} = \infty$ the modified, moving horizon control is given by

$$u^m(t) = -R_t^{-1}\, B'_t P^{-1}(t, t+T)x(t) \qquad (2.10)$$

and the computation requires the inversion of $P(t, t+T)$ at all t.

It was proved by Kalman (1960) that the 'steady-state' time-varying feedback gain $K(t, \infty)$ stabilizes, under uniform complete controllability assumptions, the time-varying system (2.1). Practically however, there is no way to obtain $K(t, \infty)$. The moving-horizon control laws are readily computable in principle, since they require solutions of recursions over a finite time-span. Even better than that, we can derive efficient gain-update algorithms that remove the necessity to re-solve for each time point the backwards Riccati equation, thus rendering the method computationally efficient.

3. General gain update recursions

Using some results from the scattering theory originally developed by Redheffer (1961) for the study of transmission-line problems, and then applied to estimation and control theory by Kailath (1979), we derive recursive update algorithms for the gain required in moving horizon controls. We refer to Redheffer (1961), Kailath (1979), Verghese *et al.* (1980), Bruckstein and Kailath (1982) for comprehensive reviews of scattering theory. The main idea of scattering theory is to embed the Riccati variable, $K(\tau, \sigma)$, that satisfies a forwards or backwards equation, into a scattering matrix $S_{\tau\sigma}$. The embedding is achieved by defining the auxiliary matrices $\Phi(\tau, \sigma)$, $\Psi(\tau, \sigma)$ and $L(\tau, \sigma)$ through the following differential equations

$$\frac{\partial}{\partial \tau} \Phi(\tau, \sigma) = \Phi(\tau, \sigma)[A_\tau - B_\tau R_\tau^{-1} B'_\tau K(\tau, \sigma)] \tag{3.1}$$

$$\frac{\partial}{\partial \tau} \Psi(\tau, \sigma) = [A'_\tau - K(\tau, \sigma) B_\tau R_\tau^{-1} B'_\tau] \Psi(\tau, \sigma) \tag{3.2}$$

$$\frac{\partial}{\partial \tau} L(\tau, \sigma) = \Phi(\tau, \sigma) B_\tau R_\tau^{-1} B'_\tau \Psi(\tau, \sigma) \tag{3.3}$$

together with the backwards Riccati equation (2.4), in which the final time has been parametrized, i.e.

$$-\frac{\partial}{\partial \tau} K(\tau, \sigma) = A'_\tau K(\tau, \sigma) + K(\tau, \sigma) A_\tau - K(\tau, \sigma) B_\tau R_\tau^{-1} B'_\tau K(\tau, \sigma) + Q_\tau \tag{3.4}$$

and appropriate final conditions at $\tau = \sigma$, these equations yield the backwards evolution of a $2n \times 2n$ scattering matrix, defined as

$$S_{\tau\sigma} = \begin{bmatrix} \Phi(\tau, \sigma) & L(\tau, \sigma) \\ K(\tau, \sigma) & \Psi(\tau, \sigma) \end{bmatrix} \quad \text{with} \quad S_{\sigma\sigma} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3.5}$$

This 'chosen final values matrix' does not display the correct starting point for the gain recursion. Since however, we also have, through scattering theory, a method to change initial or final conditions of Riccati equations to arbitrary values, we shall in the sequel concentrate on this generic case of $F_\sigma = 0$. After the gain is computed for this particular final condition the true gain can readily be obtained using a 'change of final condition' formula (see (3.11)).

Now, a two parameter matrix $S_{\tau\sigma}$ has been defined through a set of backwards differential equations together with a 'canonical' final condition, an identity matrix with dimension $2n$. In the sequel, we shall have to exploit the properties of this matrix, in particular it's behaviour as the parameter σ varies infinitesimally. In this context, we have the following basic result due to Redheffer (1961) (see also Reid 1970).

Theorem 3.1

Given that the matrix functions involved in the definition of the linear system (2.1) and the cost function (2.2) are piecewise continuous, the elements of the scattering matrix obey the following forwards differential equations

$$\frac{\partial}{\partial \sigma} \Phi(\tau, \sigma) = [A_\sigma + L(\tau, \sigma) Q_\sigma] \Phi(\tau, \sigma) \tag{3.6}$$

$$\frac{\partial}{\partial \sigma} \Psi(\tau, \sigma) = \Psi(\tau, \sigma)[A'_\sigma + Q_\sigma L(\tau, \sigma)] \tag{3.7}$$

$$\frac{\partial}{\partial \sigma} L(\tau, \sigma) = A_\sigma L(\tau, \sigma) + L(\tau, \sigma) A'_\sigma + L(\tau, \sigma) Q_\sigma L(\tau, \sigma) - B_\sigma R_\sigma^{-1} B'_\sigma \tag{3.8}$$

$$\frac{\partial}{\partial \sigma} K(\tau, \sigma) = \Psi(\tau, \sigma) Q_\sigma \Phi(\tau, \sigma) \tag{3.9}$$

the initial condition for $S_{\tau\sigma}$ at $\sigma = \tau$ being the identity matrix.

The above result together with the backwards equations defining the scattering matrix, provides the evolution of $S_{\tau\sigma}$ with respect to variations in each index, τ and σ. Now, realizing that the moving horizon gain is obtained as a submatrix of $S(t, t+T)$ (up to a change to the true final conditions) we need to derive evolution equations for the case of simultaneous variation in the parameters, according to $\tau = t$ and $\sigma = t + T$. We have the following.

Theorem 3.2

The evolution equations, for the sliding window scattering matrix, are given by

$$\frac{d}{dt} S(t, t+T) = \frac{\partial}{\partial \tau} S(\tau, \sigma)\big|_{\tau=t;\, \sigma=t+T} + \frac{\partial}{\partial \sigma} S(\tau, \sigma)\big|_{\tau=t;\, \sigma=t+T} \tag{3.10}$$

This obvious relation, essentially the chain rule for differentiation, provides a complete set of recursions for the (zero final condition) feedback gain and the auxiliary matrices. These recursions are a combination of the evolution equations yielding the variations of $S_{\tau\sigma}$ with τ and σ separately.

The initialization of the recursions of type (3.10) is done by first solving the backwards equations for an initial interval, say $(t_0, t_0 + T)$, for both the gain and the auxiliary matrices.

In order to find the true feedback gain sequence, corresponding to the sequence of final conditions F_t, we shall use the following formula, derived from a basic closure property of the Riccati equation

$$\begin{aligned} K(t, t+T\,;\; F_{t+T}) &= K(t, t+T) \\ &\quad + \Psi(t, t+T) F_{t+T} [I - L(t, t+T) F_{t+T}]^{-1} \Phi(t, t+T) \end{aligned} \tag{3.11}$$

where $K(\tau, \sigma; F_\sigma)$ denotes, as before, the gain obtained through the backwards Riccati equation with final condition F_σ.

Note that eqn. (3.11) provides the true gain by operating on the entries of $S_{t(t+T)}$ and on F_{t+T}. More about the derivations, essentially very simple when relying on scattering theory, of the above mentioned results can be found in Kailath (1979), Verghese *et al.* (1980), Bruckstein and Kailath (1982).

It is interesting to note that in the case of $F_t \to \infty$, we obtain from eqn. (3.11) the following result

$$K(t, t+T; \infty) = K(t, t+T) - \Psi(t, t+T)L^{-1}(t, t+T)\Phi(t, t+T) \tag{3.12}$$

which expresses the gain $P^{-1}(t, t+T)$ in terms of the recursively obtained sliding-window scattering matrix.

Some extensions and numerical considerations

In the case of constant final weighting in the moving horizon criterion, we can also derive recursions of the type eqn. (3.10) that provide the true gain directly, avoiding the need for a continuous final condition adjustment. Also in the case of given differential behaviour of F_t the above simplification is feasible. In all these cases the recursions turn out to have the form of eqn. (3.10), the modifications amounting to predetermined changes in the matrices appearing in the recursions with index $t+T$. This is of course not unexpected, since the changes occur at the right end boundary conditions.

It is also obvious that the same embedding can be done starting with the $P(t, t+T)$ gain, which again obeys a backwards Riccati equation, and this method will provide direct recursions for it.

The above derived algorithms, although involving a matrix twice the size of the gain matrix needed, have the potential of being computationally much more efficient than solving the backwards Riccati recursion at each point in time. (Also note that we have considerable redundancy in the scattering matrix, since $\Phi(\tau, \sigma) = \Psi'(\tau, \sigma)$ by definition.) Some numerical problems may however arise in propagating the sliding-window differential equations; the updating algorithms are likely to cumulate errors and thus the gains obtained after a large number of iterations will differ from the correct ones. The situation can be ameliorated by implementing a restart procedure at adaptively determined intervals. The idea is to compute, at intervals, the 'true' gain through the usual forwards or backwards growing-memory update algorithm and to compare the result with the gain provided by the sliding-window propagation. The time interval to the next such check procedure can then be increased or decreased according to a suitably defined measure of the difference between the 'true' and propagated gain. (Obviously, the sliding-window algorithms will be propagated with the 'true' values as initial conditions between the test intervals, i.e. will be 'restarted'.) This process will clearly determine the period of time at which a restart is necessary in order to keep the error in the gain within a predetermined bound.

Thus moving horizon control design is a computationally feasible method for stabilizing time varying systems. These ideas can obviously be applied to discrete-time moving horizon control problems (see, for example, Kwon and Pearson 1978, Bruckstein and Kailath 1982).

4. Closed-loop stability results

As pointed out earlier, it is a well known result that the feedback control law with $K(t, \infty)$ provides an asymptotically stable closed-loop system, under uniform controllability and observability assumptions. It is not obvious at this point, however, whether the sliding horizon controls can stabilize a general time-varying system.

The following simple example is quite suggestive. Let us consider a scalar, time-invariant system, $\dot{x} = ax + bu$, with the modified receding horizon cost $J_m = \int_t^{t+T} (qx^2 + ru^2)\, d\tau$. It can be shown by straightforward algebra that, the closed-loop system is stable when $T > (2a\alpha)^{-1} \ln [(\alpha+1)/(\alpha-1)]$ for $a > 0$, where $\alpha = (1 + qb^2/ra^2)^{1/2}$.

This example shows that even without terminal weight there exists a finite horizon that stabilizes the system. In the general infinite horizon case, the terminal weighting matrix plays no role and so can be arbitrarily set to zero. Since the method we describe introduces a finite, sliding horizon concept, the weighting matrix F_t, which is essentially a design parameter, plays a crucial role in determining the properties of the resulting control law. In the sequel, we shall discuss three different choices for F_t

Case 1 $$F_t = 0 \tag{4.1 a}$$

Case 2 $$F_t = \infty \tag{4.1 b}$$

Case 3 $$\left\{ F_t : \ F_t > 0, \frac{d}{dt} F_t + A'_t F_t + F_t A_t - F_t B_t R_t^{-1} B'_t F_t + Q_t \leqslant 0 \right\} \tag{4.1 c}$$

In this classification, the infinite final weighting can, of course, be considered as a particular case of (4.1 *c*). It has a special value, however, being the most useful one for practical design, as will become clear later.

The existence of a finite, though possibly large horizon for which the control law stabilizes, the system can easily be proven for the time-invariant case. We have the following result.

Theorem 4.1

If the pair $\{A, B\}$ is controllable and we have $F = 0$, $Q > 0$, $R > 0$, then there exists a finite horizon T, such that the moving horizon control law eqn. (2.6) stabilizes the system.

Proof

The closed-loop system is given by

$$\frac{d}{dt} x(t) = [A - BR^{-1}B'K_T]x(t)$$

where K_T is obtained from eqn. (2.8). Let $\bar{K}$ be the solution of the algebraic Riccati equation, i.e. the steady-state feedback gain. Let $K_T{}^e = \bar{K} - K_T$. Then the closed-loop system can be written as

$$\frac{d}{dt} x(t) = [A - BR^{-1}B'\bar{K}]x(t) + BR^{-1}B'K_T{}^e x(t)$$

Now, since $A - BR^{-1}B'\bar{K}$ is a stable matrix, it is sufficient to show that the 'perturbation term' $\|BR^{-1}B'K_T{}^e x\|/\|x\|$ can be made arbitrarily small for some T. But it is known that $K_{t_1} \leqslant K_{t_2}$ for $t_1 \leqslant t_2$ and that $K_T \to \bar{K}$ as $T \to \infty$, thus

$$\frac{\|BR^{-1}B'K_T{}^e x\|}{\|x\|} \to 0 \quad \text{as } T \to \infty$$

and therefore, there exists a finite T such that the perturbation is arbitrarily close to zero. This completes the proof. □

The existence of a possibly very large horizon for which the modified control yields a stable closed-loop system, for the general time-varying case, can be shown using a similar approach. Intuitively, it is clear that, for a very large T, the value of $K(t, t+T\,;\ F) \to K(t, \infty)$ to an arbitrary degree. This provides a control that differs only slightly from the steady-state feedback law, which is known to stabilize the time-varying system.

The case of zero final weighting leads thus to generally large horizons and also problems in determining a suitable depth T. The case of $F_t = \infty$ for which we actually give a method for choosing the horizon depth, that turns out to be connected to the controllability properties of the system to be stabilized, is therefore of greater practical interest. In order to state the results in this case we recall the following definition.

Definition

The pair $\{A(t), B(t)\}$ is uniformly completely controllable if for some $\delta > 0$ the following inequalities hold for all t

$$\alpha_1 I \leqslant W(t, t+\delta) \leqslant \alpha_2 I \tag{4.2}$$

$$\|\phi(t_1, t_2)\| \leqslant \gamma |t_1 - t_2| \tag{4.3}$$

In these expressions, $\phi(\tau, \sigma)$ stands for the state transition matrix of $A(t)$, α is a positive constant, $\gamma(\cdot)$ maps R into R and is bounded on bounded intervals, and $W(\tau, \sigma)$ is the controllability matrix defined as

$$W(t_1, t_2) = \int_{t_1}^{t_2} \phi(t_1, \sigma) B_\sigma B'_{\sigma\phi'}(t_1, \sigma)\, d\sigma \tag{4.4}$$

We now have the following result on the stability of moving horizon control laws with $F_t = \infty$, which was first proved by Kwon and Pearson (1977).

Theorem 4.2

If the pair $\{A_t, B_t\}$ is uniformly completely controllable, and if $0 \leqslant Q_t \leqslant \alpha_4 I$ and $\alpha_5 I \leqslant R_t \leqslant \alpha_6 I$, then for any $T > \delta$, the moving horizon control law (2.10) stabilizes the system (2.1).

Now, in addition to the previous particular cases of final weighting, a somewhat more general class of final weighting sequences will be discussed. The most general case of an arbitrary sequence F_t is extremely difficult to handle due to a lack of known monotonicity properties for solutions of Riccati equations. Therefore, we shall investigate the class of sequences defined by (4.1 c). It may be that with further effort these results can be extended to more general situations, as has been done for time-invariant systems (see, for example, Willems 1971, Wonham 1968).

In order to prove the main result, some properties of Riccati equations have to be established first. These are summarized in the following lemma (see also Reid 1970).

Lemma 4.1

(1) If the matrix F_t belongs to the class defined by (4.1 c), then $K(\tau, \sigma)$, the solution of the backward Riccati equation (2.4), satisfies the following inequalities

$$K(\tau, \sigma_1 ;\ F_{\sigma_1}) \geqslant K(\tau, \sigma_2 ;\ F_{\sigma_2}) \quad \text{for } \tau \leqslant \sigma_1 \leqslant \sigma_2 \tag{4.5}$$

(2) If the pair $\{A_t, B_t\}$ is uniformly completely controllable and for all t, $\alpha_3 I \leqslant Q_t \leqslant \alpha_4 I$ and $\alpha_5 I \leqslant R_t \leqslant \alpha_6 I$ then for any T such that $\delta \leqslant T \leqslant \infty$ there exist positive constants α_7 and α_8 such that

$$\alpha_7 I \leqslant K(t, t+T ;\ F_{t+T}) \leqslant \alpha_8 I \tag{4.6}$$

Proof

(1) We shall derive this result using some monotonic properties of Riccati equations. Consider two backwards Riccati equations as follows

$$-\frac{d}{dt} M(t) = M(t)A_t + A'_t M(t) - M(t)B_t R_t^{-1} B'_t M(t) + Q_t^1 \tag{4.7 a}$$

$$-\frac{d}{dt} N(t) = N(t)A_t + A'_t N(t) - N(t)B_t R_t^{-1} B'_t N(t) + Q_t^2 \tag{4.7 b}$$

with final conditions given respectively by

$$M(t_f) = F_1 \quad \text{and} \quad N(t_f) = F_2$$

Now it is readily realized that if $F_1 = F_2$, and $Q_t^1 \geqslant Q_t^2$, then $M(t) \geqslant N(t)$ for all $t \leqslant t_f$. Similarly, we can show that provided $Q_t^1 = Q_t^2$, $F_1 \geqslant F_2$ implies $M(t) \geqslant N(t)$ at all $t \leqslant t_f$. These inequalities immediately provide (4.5) since by choosing $K(\sigma_2, \sigma_2 ;\ F_{\sigma_2}) = F_{\sigma_2}$ we shall have $K(\sigma_1, \sigma_2 ;\ F_{\sigma_2}) \leqslant F_{\sigma_2}$ by (4.1 c) and therefore $K(t, \sigma_1 ;\ F_{\sigma_1}) \geqslant K(t, \sigma_2 ;\ F_{\sigma_2})$ is implied by the fact that $K(\sigma_1, \sigma_1 ;\ F_{\sigma_1}) = F_{\sigma_1} \leqslant K(\sigma_1, \sigma_2 ;\ F_{\sigma_2})$.

(2) The result of the second part of this lemma may be obtained by a slight modification of the arguments given by Kalman (1960), Anderson and Moore (1969), Bucy (1972), Canales (1970) and the definition of uniform complete controllability. □

Let us now give an interpretation to condition (4.1 c) : note that this condition implies that the matrix F_t satisfies the following differential equation (of course under the assumption that its evolution is differentiable)

$$-\frac{d}{dt} F_t = F_t A_t + A'_t F_t - F_t B_t R_t^{-1} B'_t F_t + Q_t + H_t H'_t \tag{4.8}$$

for some matrix sequence H_t. Let $K(t, t_f ;\ F_{t_f})$ be the solution of the backwards equation (2.4) with final condition F_{t_f}. Then we have

$$F_t \geqslant K(t, t_f ;\ F_{t_f}) \quad \text{for all } t \leqslant t_f \tag{4.9}$$

and thus we can state that the steady-state solution for the gain $K(t, \infty)$ has to be a lower bound on the matrix sequences of class (4.1 *c*). We are now in a position to prove the following result.

Theorem 4.3

If F_t is a matrix sequence of class (4.1 *c*), R_t and Q_t satisfy the conditions of Lemma 4.1 and the pair $\{A_t, B_t\}$ is uniformly completely controllable, then for any fixed T such that $\delta \leqslant T \leqslant \infty$, the control (2.6) yields a closed-loop system that is uniformly asymptotically stable.

Proof

Let us consider the adjoint of the closed-loop system, with the feedback gain given by (2.6), where the final weighting sequence is of class (4.1 *c*)

$$\frac{d}{dt} x_a(t) = -\Lambda'_t x_a(t) \tag{4.10}$$

Here $\Lambda_t = A_t - B_t R_t^{-1} B'_t K_t$ where $K_t = K(t, t+T\ ;\ F_{t+T})$. Further, consider an associated scalar-valued function

$$V(x_a, t) = x'_a(t) K_t^{-1} x_a(t)$$

which, by part (2) of Lemma 4.1, satisfies

$$\alpha_9 \|x_a\|^2 \leqslant V(x_a, t) \leqslant \alpha_{10} \|x_a\|^2$$

Therefore, this function is a positive definite function of the adjoint system state. Now, the asymptotic stability of the original closed-loop system is guaranteed if the adjoint system state vector is exponentially increasing (i.e. the adjoint system is asymptotically unstable). But we can show that

$$\begin{aligned} \frac{d}{dt} V(x_a, t) &= x'_a \left\{ -A_t K_t^{-1} - K_t^{-1} A'_t + 2B_t R_t^{-1} B'_t + \frac{d}{dt} K_t^{-1} \right\} x_a \\ &= x'_a \left\{ B_t R_t^{-1} B'_t + K_t^{-1} Q_t K_t^{-1} + \frac{\partial}{\partial \sigma} K^{-1}(t, \sigma\ ;\ F_\sigma)|_{\sigma = t+T} \right\} x_a \end{aligned}$$

and using the result of part (1), Lemma 4.1, on the monotonic properties of the gain with respect to changes in the second parameter, we obtain

$$\frac{d}{dt} V(x_a, t) \geqslant x'_a B_t R_t^{-1} B'_t x_a \tag{4.11}$$

Now, let $\Phi_k(\tau, \sigma)$ be the transition matrix of the closed-loop system $\dot{x} = \Lambda_t x$. From eqn. (4.11) it follows that

$$V(x_a(t_1), t_1) - V(x_a(t_0), t_0)$$

$$\geqslant x'_a(t_0) \left[\int_{t_0}^{t_1} \Phi_k(t_0, t) B_t R_t^{-1} B_t \Phi'_k(t_0, t)\, dt \right] x_a(t_0) \tag{4.12}$$

$$\geqslant \alpha_{11} \|x_a(t_0)\|^2 \quad \text{for } t_1 \geqslant t_0 + \delta \tag{4.13}$$

for some positive constant α_{11}. The inequality (4.13) results from the fact that the closed-loop system is uniformly controllable (Anderson and Moore 1969), since there exists some positive constant α_{12} such that

$$\int_t^{t+\delta} \|R_\xi^{-1} B'_\xi K_\xi\|^2 \, d\xi \leqslant \alpha_{12}$$

But, eqn. (4.12) implies that the adjoint system increases exponentially, which in return guarantees the asymptotic stability of the original system with the feedback under consideration. This completes the proof. □

This theorem indicates that there exists a whole class of terminal weighting matrix sequences that, with a horizon even slightly greater than the controllability interval δ, yield stabilizing control laws. From these results it is also conjectured that large horizons are necessary if the final weighting matrix is small, whereas for sequences of large final weightings the stabilizing horizon approaches the controllability interval δ.

The results simplify somewhat for the case of time-invariant systems. The class (4.1 *c*), of constant final weighting matrices, is readily seen to be the set of final weighting matrices satisfying $F \geqslant K_\infty$, the solution of the algebraic Riccati equation associated with the control problem.

5. Some further topics and results

5.1. *Cost incurred by moving horizon laws*

The moving horizon control dynamically redefines the performance criterion and, therefore, it clearly does not optimize any overall criterion. Since the standard cost of the type eqn. (2.2) is an accepted measure for the performance of a control law, it is useful to determine bounds on the cost incurred by the modified control laws. Indeed, we can prove the following theorem.

Theorem 5.1

The standard quadratic cost incurred by the sliding horizon control has the following bound

$$\int_{t_i}^{\infty} [x'(t)Q_t x(t) + u'(t)R_t u(t)] \, dt \leqslant x'(t_i)K(t_i, t_i + T \,;\; F_{t_i+T})x(t_i) \tag{5.1}$$

Proof

The quadratic cost for a linear feedback control of the type eqn. (2.3) is given by $x'(t_i)M(t_i, t_f)x(t_i)$, where $M(t, t_f)$ obeys the following backwards recursion

$$-\frac{d}{dt} M(t, t_f) = \Lambda'_t \, M(t, t_f) + M(t, t_f)\Lambda_t$$
$$+ K(t, t+T \,;\; F_{t+T})B_t R_t^{-1} B'_t \, K(t, t+T \,;\; F_{t+T}) + Q_t$$

with the final condition $M(t_f, t_f) = F_{t_f}$. From eqn. (3.10) we have

$$-\frac{d}{dt} K(t, t+T\,;\; F_{t+T}) = \Lambda'_t K(t, t+T\,;\; F_{t+T}) + K(t, t+T\,;\; F_{t+T})\Lambda_t + K(t, t+T\,;\; F_{t+T}) B_t R_t^{-1} B_t K(t, t+T\,;\; F_{t+T}) + Q_t - \frac{\partial}{\partial\sigma} K(t, \sigma\,;\; F_\sigma)\big|_{\sigma=t+T} \tag{5.2}$$

Now define $E(t) = M(t, t_f) - K(t, t+T\,;\; F_{t+T})$, which obviously satisfies the following equation

$$-\frac{d}{dt} E(t) = \Lambda'_t E(t) + E(t)\Lambda_t + \frac{\partial}{\partial\sigma} K(t, \sigma\,;\; F_\sigma)\big|_{\sigma=t+T}$$

with boundary condition at t_f given by $F_{t_f} - K(t_f, t_f+T\,;\; F_{t+T})$. Integrating this equation and taking into consideration the fact that the last term is negative (by Lemma 4.1), we obtain

$$E(t_i) \leqslant \Phi'(t_f, t_i)[F_{t_f} - K(t_f, t_f+T\,;\; F_{t_f+T})]\Phi(t_f, t_i) \tag{5.3}$$

Since the closed-loop system was proved asymptotically stable, we shall have $E(t_i) \to 0$ as $t_f \to \infty$. Therefore, $M(t, \infty) \leqslant K(t, t+T\,;\; F_{t+T})$ which establishes the desired result. □

Further work on moving horizon controls is still needed to see whether it also inherits more of the desirable features of ' optimal ' feedback controls, as for example, robustness and good sensitivity properties.

5.2. *A structurally dual state estimator*

Since it is well known that there exists a duality between optimal linear regulator problems and optimal state estimation, one may ask what state estimation filter is the dual of the moving horizon control. The answer is indeed simple. Structurally, the dual system is a Kalman filter with a sliding window gain defined as the solution of a forward Riccati equation over a fixed length interval $(t-T, t)$. This estimator is a suboptimal state reconstruction filter and can be analysed by the methods employed above in order to establish stability results. Also, obviously, the gain update equations can be derived in a similar way.

An important fact to realize however, is that this filter does not provide the ' best ' estimate of the state given the observations over the sliding interval. It is not a solution to the limited memory state-space filtering problem, as posed for example in Jazwinski (1968), since it does not ' completely forget ' data beyond $t-T$. This can be easily seen from the fact that the filter has infinite impulse response. Scattering theory does however provide a solution to this problem as well. The solution involves the application of an idea similar to the one that led to efficient gain update formulae to an extended scattering matrix (Bruckstein and Kailath 1982).

6. Concluding remarks

A general and computationally feasible method of stabilizing time-varying linear systems through state-feedback was presented. A variant of the method, the case of infinite final weighting, was previously analysed by Kwon and Pearson (1977), and the system stabilizing property of the resulting control law were established. This paper further generalizes the moving horizon method to a whole class of final weighting matrices and also provides explicit gain update algorithms which render the method more efficient computationally.

Several extensions of the method are possible. We could, for example, deal with a time varying horizon depth T_t, and it should be clear that the approach presented in § 3 easily yields the gain update algorithms for this case too. One might wish to change the horizon depth to adapt to varying controllability properties of the system under control. The case of discrete time systems can easily be treated within the same framework, the results being somewhat more involved algebraically but not conceptually (see, for example, Kwon and Pearson 1978, Bruckstein and Kailath 1982).

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