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# Some Matrix Factorization Identities for Discrete Inverse Scattering*

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ABSTRACT

This note analyzes the relationships between several inverse-scattering methods and points out that all classical approaches implicitly construct a (lower-upper) triangular factorization of a given positive definite Toeplitz matrix. It is shown that the various inversion methods implicitly obtain the factorization by solving different nested sets of linear equations and expressing the factors in terms of the solutions obtained. The fact that the triangular factorization of a matrix with nonzero leading minors is unique immediately yields various identities, e.g., involving the so-called "central mass" sequence, that were derived in the literature with considerably more manipulation. In fact, our basic factorization results are somewhat more general, since they do not require the Toeplitz assumption.

## 1. INTRODUCTION

In a recent paper, Caflisch [1] presented the solution to an inverse problem for discrete transmission lines. His approach closely follows the solution of Gopinath and Sondhi [2] for the corresponding continuous transmission-line problem.

The inverse problem for discrete transmission lines is the following. A semiinfinite portion of a passive transmission line that has a piecewise

*This work was supported in part by the U.S. Army Research Office under Contract DAAG29-79-C-0215, and by the Air Force Office of Scientific Research under Contract AF49-620-79-C-0058. Professor Thomas Kailath also gratefully acknowledges the support provided by an Erna and Jakob Michael Visiting Chair in Theoretical Mathematics at the Weizmann Institute of Science, Rehovot, Israel, during Spring Quarter, 1984.




52 Vanderbilt Ave., New York, NY 10017 0024-3795/86/$3.50

constant impedance profile over $[0,\infty)$ defined by

$$Z(x)=Z_i \quad \text{for} \quad x\in\left[\frac{i}{2},\frac{i+1}{2}\right] \qquad (Z_0=1) \tag{1.1}$$

is probed by exciting it with a current impulse $I(0,t)=\delta(t)$ applied at $x=0$ and measuring the voltage response $V(0,t)$ at that point. The voltage response to a current impulse has the form

$$V(0,t)=\delta(t)+2\sum_1^\infty R_i\delta(t-i). \tag{1.2}$$

Taking $z$-transforms, the input impedance will be

$$\frac{V(0,z)}{I(0,z)}=1+2\sum_1^\infty R_i z^{-i}, \tag{1.3}$$

and the passivity of the line means that this must be a positive real function, i.e., it must be analytic in $|z|>1$ and also obey

$$\mathrm{Re}\left\{1+2\sum_1^\infty R_i z^{-i}\right\}>0, \qquad |z|\geqslant 1. \tag{1.4}$$

It follows (by the Riesz-Herglotz theorem; see e.g. Rudin [3]) that $\{1,R_1,R_2,\dots,\}$ is a positive semidefinite sequence, i.e., that for all $n$, the Toeplitz matrix

$$\mathbf{R}_n=\begin{bmatrix} 1 & R_1 & & & & R_n \\ R_1 & \cdot & \cdot & & & \\ & \cdot & \vdots & \vdots & \cdot & \\ & & & \cdot & \cdot & R_1 \\ R_n & & & & R_1 & 1 \end{bmatrix} \tag{1.5}$$

obeys

$$\det\mathbf{R}_n\geqslant 0, \qquad n=0,1,2,\dots. \tag{1.6}$$

If $\det \mathbf{R}_N = 0$, then so will $\det \mathbf{R}_{n+i}$, $i \geqslant 0$, and therefore it is reasonable to restrict ourselves to the range of $n$ for which $\det \mathbf{R}_n > 0$. This will be assumed henceforth. The inverse transmission-line problem is to determine the impedance profile $\{Z_1, Z_2, Z_3, \dots\}$ from the data sequence $\{1, R_1, R_2, \dots\}$.

A key element of the approach of Sondhi and Gopinath, and therefore of Caflisch's solution to the inversion problem, is to define and use the so-called "central mass" function (sequence, in the discrete case) of M. G. Krein [4] (see also Landau [5]). In the discrete case this sequence is defined as follows. Consider the family of Toeplitz equations

$$\begin{bmatrix} 1 & R_1 & & & R_n \\ R_1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & R_1 \\ R_n & & & R_1 & 1 \end{bmatrix} \begin{bmatrix} c_n(0) \\ c_n(1) \\ \vdots \\ c_n(n) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad n = 0, 1, 2, \dots, N. \tag{1.7}$$

Then the "central mass sequence" associated with the data $\{R_1, R_1, \dots, R_N\}$, or the corresponding Toeplitz matrix, is defined as the ordered set

$$\left\{ \rho_n^{-1} = \sum_{i=0}^{n} C_n(i),\ n = 0, 1, 2, \dots, N \right\}. \tag{1.8}$$

The solution to the inverse transmission-line problem is then obtained by deducing the identity

$$\frac{1}{\rho_n} = \sum_{i=0}^{n} \frac{1}{Z_i} \tag{1.9}$$

where $\{Z_i\}$ is the sequence defining the impedance profile of the transmission line. Caflisch's derivation of this result is based on some rather detailed energy-balance calculations, closely paralleling the arguments of Gopinath and Sondhi for the continuous problem (see [1] and [2]).

It is known that there are several other methods of solving the inverse problem (see, e.g., [6]–[9]). In particular, one of these is based on the now fairly widely known Levinson algorithm (see, e.g., [10]–[12]) for recursively solving the family of Toeplitz equations

$$\begin{bmatrix} 1 & R_1 & & & R_n \\ R_1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & R_1 \\ R_n & & & R_1 & 1 \end{bmatrix} \begin{bmatrix} a_n(0) \\ a_n(1) \\ \cdot \\ \cdot \\ \cdot \\ a_n(n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \qquad n = 0,1,2,\dots,N. \tag{1.10}$$

If we define the polynomials

$$A_n(z) = z^n a_n(n) + z^{n-1} a_n(n-1) + \cdots + z a_n(1) + a_n(0), \tag{1.11a}$$

$$A_n^{\#}(z) = z^n a_n(0) + z^{n-1} a_n(1) + \cdots + z a_n(n-1) + a_n(n), \tag{1.11b}$$

then the Levinson recursion is

$$\begin{aligned} \begin{bmatrix} A_n(z) \\ A_n^{\#}(z) \end{bmatrix} &= \frac{1}{1-k_n^2} \begin{bmatrix} 1 & -k_n \\ -k_n & 1 \end{bmatrix} \begin{bmatrix} zA_{n-1}(z) \\ A_{n-1}^{\#}(z) \end{bmatrix} \\ &= \prod_{i=1}^{n} \frac{1}{1-k_i^2} \begin{bmatrix} z & -k_i \\ -zk_i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \end{aligned} \tag{1.12}$$

where the quantities $k_n$, needed for the propagation of the recursions are computed via

$$k_n = \sum_{i=1}^{n} R_i a_{n-1}(i-1) \tag{1.13}$$

and are known as the *reflection coefficients*. Derivations of the Levinson algorithm can be found in many places; see for example [10]–[12].

A simple analysis of discrete transmission lines shows (see, e.g., [8]–[9]) that the reflection coefficients are related to the impedance profile by the formulas

$$k_n = \frac{Z_n - Z_{n-1}}{Z_n + Z_{n-1}}, \qquad Z_n = \frac{1+k_n}{1-k_n} Z_{n-1}. \tag{1.14}$$

Therefore,

$$Z_n = \prod_{i=1}^{n} \frac{1+k_i}{1-k_i}. \tag{1.15}$$

The solution of the inverse problem by this method is to first find the $\{k_i\}$ sequence from the $\{R_i\}$ via (1.12) and (1.13) and then to use (1.15) for impedance profile reconstruction.

Another, closely related approach to the inverse problem can be based on solutions of the family of equations

$$\begin{bmatrix} 1 & R_1 & & & R_n \\ R_1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & R_1 \\ R_n & & & R_1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_n(0) \\ \lambda_n(1) \\ \vdots \\ \lambda_n(n) \end{bmatrix} = \begin{bmatrix} \omega^n \\ \vdots \\ \omega^2 \\ \omega \\ 1 \end{bmatrix}, \qquad n = 0, 1, 2, \ldots, N, \tag{1.16}$$

which are continuous analogs of equations used in radiative-transfer theory (see, e.g., Kagiwada and Kalaba [13]). Note that setting $\omega = 1$ and $\omega = 0$ leads to the equations (1.7) and (1.10).

We shall show that in fact any of the sets of solutions for matrix equations $\mathbf{R}_n \boldsymbol{\Gamma}_n = \mathbf{F}_n$, with right-hand sides that have some type of nesting property, will also implicitly solve the inverse problem.

All the different methods are of course related, since they ultimately solve the same problem. Without considering the details of the various inversion procedures, the aim of the present paper is to point out that the key to understanding the relationships between these methods is to note that each family of equations of the type (1.16) implicitly constructs a Cholesky factorization for the inverse of the coefficient matrix $\mathbf{R}_N$, where the triangular factors are expressed in terms of the families of solutions $\{c_n(\cdot)\}, \{a_n(\cdot)\}, \{\lambda_n(\cdot)\}$. But the triangular factors of $\mathbf{R}_N$, a positive definite Toeplitz matrix in the inverse problem, are *unique*, and therefore the factorizations obtained for $\mathbf{R}_N^{-1}$, expressed in terms of the different families of solutions, must be identical. This simple fact easily leads to many of the identities used and often rederived in the literature of inverse scattering.

In the course of our derivation, we find that the basic formulas for triangular factorization do not depend upon the Toeplitz nature of $\mathbf{R}_N$, but in fact apply to any symmetric matrix with nonzero leading minors. The results of Section 2 will be derived under this assumption, and we shall only return to the Toeplitz assumption in the latter part of Section 3, when we explicitly consider the classical inverse scattering problem.

## 2. TRIANGULAR DECOMPOSITIONS OF SYMMETRIC MATRICES WITH NONZERO LEADING MINORS

It was perhaps Burg [11] who first pointed out, in connection with the Levinson algorithm, that from successive solutions of (1.4) one could immediately write down a triangular factorization of $\mathbf{R}_N^{-1}$. Moreover, this result does not need the fact that $\mathbf{R}_N$ is Toeplitz; it will be enough that $\mathbf{R}_N$ be symmetric and have nonzero leading minors (see, e.g., Kailath, Vieira and Morf [12]). *This will be true for all the results presented in this section.* Indeed, note that by (1.4) we have

$$\mathbf{R}_N \begin{bmatrix} a_0(0) & a_1(0) & \cdots & a_N(0) \\ 0 & a_1(1) & \cdots & a_N(1) \\ 0 & 0 & \cdots & a_N(2) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_N(N) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \times & 1 & 0 & 0 & \cdots & 0 \\ \times & \times & 1 & 0 & \cdots & 0 \\ \times & \times & \times & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \times & \times & \times & \times & \cdots & 1 \end{bmatrix}, \tag{2.1}$$

where $\times$'s denote entries whose exact, generally nonzero, values are not important at present. For compactness we write (2.1) as

$$\mathbf{R}_N \mathbf{a}_N = \mathbf{L}_N \tag{2.2}$$

and also define

$$\mathbf{A}_N = \mathbf{a}_N D_a^{-1}, \qquad D_a = \operatorname{diag}\{a_i(n)\} \tag{2.3}$$

so that $\mathbf{A}_N$ is lower-triangular with ones on the diagonal. Then we can write

$$\mathbf{R}_N^{-1} = \mathbf{A}_N D_a \mathbf{L}_N^{-1}. \tag{2.4}$$

which is in the so-called UDL (upper-diagonal-lower) factorization form. But then, the symmetry of $\mathbf{R}_N$ and $\mathbf{R}_N^{-1}$ and the uniqueness of the UDL decomposition (see, e.g., Strang [14]) show that

$$\mathbf{L}_N^{-1} = \mathbf{A}_N^T, \qquad \text{or} \quad \mathbf{R}_N^{-1} = \mathbf{A}_N D_a \mathbf{A}_N^T. \tag{2.5}$$

Let us now do a similar exercise for a system with RHS as in (1.16). Stacking the solutions of (1.16) for increasing values of $n$, we obtain

$$\mathbf{R}_N \begin{bmatrix} \lambda_0(0) & \lambda_1(0) & \lambda_N(0) \\ 0 & \lambda_1(1) & \lambda_N(1) \\ 0 & 0 & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_N(N) \end{bmatrix} = \begin{bmatrix} 1 & \omega & \omega^2 & \cdots & \omega^N \\ \times & 1 & \omega & \ddots & \vdots \\ \times & \times & 1 & \ddots & \omega^2 \\ \vdots & \vdots & \vdots & \ddots & \omega \\ \times & \times & \times & \cdots & 1 \end{bmatrix}. \quad (2.6)$$

The LHS of the above equation is not yet lower triangular. However, this can be easily achieved by a right multiplication of both sides of (2.6) by the matrix

$$\boldsymbol{\Delta}_\omega = \begin{bmatrix} 1 & -\omega & & & \\ & 1 & \cdot & & \\ & & \ddots & \cdot & \\ & & & \cdot & -\omega \\ & & & & 1 \end{bmatrix}. \quad (2.7)$$

The result can be written, defining $\Lambda_n$ as the $(N+1)$-dimensional column vector $[\lambda_n(0), \ldots, \lambda_n(n), 0, \ldots, 0]$, as follows:

$$\mathbf{R}_N[\ \cdots \quad \boldsymbol{\Lambda}_n - \omega\boldsymbol{\Lambda}_{n-1} \quad \cdots\ ] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \times & \times & 0 & \cdots & 0 \\ \times & \times & \times & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \times & \times & \times & \cdots & \times \end{bmatrix} = \tilde{\mathbf{L}}_N. \quad (2.8)$$

This gives the decomposition

$$\mathbf{R}_N^{-1} = \begin{bmatrix} \lambda_0(0) & \lambda_1(0) - \omega\lambda_0(0) & \cdots & \lambda_N(0) - \omega\lambda_{N-1}(0) \\ & \lambda_1(1) & \cdots & \lambda_N(1) - \omega\lambda_{N-1}(1) \\ & & \ddots & \vdots \\ & & & \lambda_N(N) \end{bmatrix} \tilde{\mathbf{L}}_N^{-1}. \quad (2.9)$$

Therefore, the columns of U in the standard UDL decomposition of $\mathbf{R}_N^{-1}$ are given by

$$U_n = \frac{1}{\lambda_n(n)}[\Lambda_n - \omega\Lambda_{n-1}], \qquad \Lambda_{-1} \equiv 0. \tag{2.10}$$

The computation of the elements of the diagonal matrix $D = \text{diag}\{d_0, \dots, d_N\}$ takes a few more steps, which we relegate to the Appendix. The final result is

$$d_n = \frac{\lambda_n^2(n)}{M_\omega(n) - M_\omega(n-1)\omega^2} \tag{2.11}$$

where

$$M_\omega(n) = \langle [\lambda_n(0), \dots, \lambda_n(n)], [\omega^n, \omega^{n-1}, \dots, \omega, 1] \rangle \tag{2.12}$$

with $\langle \cdot, \cdot \rangle$ denoting the inner product of two vectors. By comparison with (1.7)–(1.8), we shall call the set $\{M_\omega(n),\ n = 0, 1, 2, \dots, N\}$ a "generalized mass sequence."

It should be noted that we have not used the Toeplitz structure of $\mathbf{R}_N$ to obtain (2.10)–(2.12). We have thus found expressions giving the UDL decomposition of the inverse of any symmetric matrix $\mathbf{R}_N$ (with nonzero leading minors) in terms of solutions of equations of the type (1.16), which includes the interesting special cases (1.10) for $\omega = 0$ and (1.7) for $\omega = 1$. For the latter case, we have with explicit specialization

$$d_n = c_n^2(n)\left(\rho_n^{-1} - \rho_{n-1}^{-1}\right)^{-1}, \tag{2.13}$$

where $c_n(n)$ and $\rho_n$ were defined by (1.7)–(1.8).

## 3. SOME FORMULAS FOR INVERSE SCATTERING

The uniqueness of the UDL decomposition immediately yields some useful identities. Comparing the diagonal entries for the cases $\omega = 0$ and $\omega = 1$ yields [cf. (2.3) and (2.13)]

$$a_n^{-1}(n) = \frac{1}{c_n^2(n)}\left(\rho_n^{-1} - \rho_{n-1}^{-1}\right), \tag{3.1}$$

from which we obtain

$$\rho_n^{-1} = \rho_{n-1}^{-1} + a_n^{-1}(n)c_n^2(n) = 1 + \sum_{l=1}^{n} a_l^{-1}(l)c_l^2(l). \tag{3.2}$$

From the uniqueness of the U (upper factor) matrix we also have that

$$\frac{c_n(i) - c_{n-1}(i)}{c_n(n)} = \frac{a_n(0)}{a_n(n)} = \frac{\lambda_n(i) - \omega\lambda_{n-1}(i)}{\lambda_n(n)} \tag{3.3}$$

Next we shall use a simple adjoint lemma that relates the solutions for different right-hand sides.

Lemma. *If* $\mathbf{R}_N$ *is symmetric and we have the solutions of the equations*

$$\mathbf{R}_N \mathbf{X}_i = \mathbf{Y}_i \qquad \text{for} \quad i = 1,2$$

*then*

$$\langle \mathbf{X}_1, \mathbf{Y}_2 \rangle = \langle \mathbf{X}_2, \mathbf{Y}_1 \rangle. \tag{3.4}$$

*Proof.* $\langle \mathbf{X}_1, \mathbf{Y}_2 \rangle = \langle \mathbf{X}_1, \mathbf{R}_N \mathbf{X}_2 \rangle = \langle \mathbf{R}_N \mathbf{X}_1, \mathbf{X}_2 \rangle = \langle \mathbf{Y}_1, \mathbf{X}_2 \rangle = \langle \mathbf{X}_2, \mathbf{Y}_1 \rangle.$ ∎

Now from this lemma for the pair of equations (1.16) and (1.10) (with Toeplitz $\mathbf{R}_N$), we immediately have that

$$\left\langle [a_n(0), \ldots, a_n(n)], [\omega^n \omega^{n-1}, \ldots, 1] \right\rangle = \lambda_n(n). \tag{3.5}$$

In particular, choosing $\omega = 0$, we get

$$\sum_{i=0}^{n} a_n(i) = c_n(n). \tag{3.6}$$

But from the Levinson recursions, it follows by setting $z = 1$ in (1.12) that

$$\sum_{i=0}^{n} a_n(i) = A_n(1) = \prod_{i=1}^{n} \frac{1}{1+k_i}. \tag{3.7}$$

Also, by setting $z=0$ in (1.12) we obtain

$$a_n(n) = A_n^{\#}(0) = \prod_{i=1}^{n} \frac{1}{1-k_i^2}. \tag{3.8}$$

Using the formula (3.2) now yields

$$\rho_n^{-1} = \rho_{n-1}^{-1} + \left( \prod_{i=1}^{n} \frac{1}{1+k_i} \right)^2 \left( \prod_{i=1}^{n} \frac{1}{1-k_i^2} \right)^{-1} = \rho_{n-1}^{-1} + \prod_{i=1}^{n} \frac{1-k_i}{1+k_i}, \tag{3.9}$$

so that

$$\rho_n^{-1} = 1 + \sum_{l=1}^{n} \prod_{i=1}^{l} \frac{1-k_i}{1+k_i}. \tag{3.10}$$

Recalling the relation (1.15) between impedances and reflection coefficients, we have

$$\rho_n^{-1} = 1 + \sum_{i=1}^{n} Z_i^{-1} = \sum_{i=0}^{n} Z_i^{-1}, \tag{3.11}$$

the result that was used by Caflisch for the reconstruction of the impedance profile. Dickinson [15] used a different approach to derive the relation (3.10), in the framework of computing likelihood ratios for a discrete pulse detection problem in stationary Gaussian noise. In [16] he also noted the connection of Caflisch's work on inverse scattering to the classical Levinson algorithm for inverting Toeplitz matrices. In [17], we have given a derivation using energy conservation and causality constraints on discrete transmission lines to derive triangular factorization and Christoffel-Darboux identities for Toeplitz and related matrices.

## 4. CONCLUDING REMARKS

The above development shows that at the heart of the traditional approaches to inverse problems lies the triangular factorization of the matrix $\mathbf{R}_N^{-1}$. We further note that such a factorization can be obtained from the solution of any *nested* set of linear equations with coefficient matrix $\mathbf{R}_N$

$$\mathbf{R}_n \begin{bmatrix} \gamma_n(0) \\ \gamma_n(1) \\ \vdots \\ \gamma_n(n) \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}, \qquad n = 0,1,2,\dots,N \tag{4.1}$$

as shown in Equations (A.12)–(A.19) of the Appendix.

When the coefficient matrix $\mathbf{R}_N$ is Toeplitz (or has some related structure), this structure can be exploited to get further results and also to solve the nested set of equations of the form (4.1) using a fast algorithm of $O(N^2)$ operations.

We also note that the above procedure for factoring $\mathbf{R}_N$, with the use of the vector $[1,1,\dots,1]$ as LHS, is a discrete analog of a method proposed by Krein for solving integral equations [4]. In fact, the formulas derived in the Appendix are the discrete (and slightly generalized) counterparts of the Toeplitz operator factorization results obtained via the Krein method (see Gohberg and Krein [4, pp. 189–205]).

## APPENDIX

Consider the equation (2.6), for any symmetric matrix $\mathbf{R}_N$ with nonzero leading minors:

$$\mathbf{R}_N \begin{bmatrix} \lambda_0(0) & \lambda_1(0) & \lambda_N(0) \\ 0 & \lambda_1(1) & \lambda_N(1) \\ 0 & 0 & \lambda_N(2) \\ \vdots & \vdots & \vdots \\ 0 & 0 & \lambda_N(N) \end{bmatrix} = \begin{bmatrix} 1 & \omega & \omega^2 & \cdots & \omega^N \\ \times & 1 & \omega & \ddots & \vdots \\ \times & \times & 1 & \ddots & \omega^2 \\ \vdots & \vdots & \vdots & \ddots & \omega \\ \times & \times & \times & \cdots & 1 \end{bmatrix}. \tag{A.1}$$

Multiplying both sides by the matrix $\mathbf{\Delta}_\omega$ defined in (2.7), it is easy to see that

$$\mathbf{R}_N \begin{bmatrix} \mathbf{\Lambda}_0 & \mathbf{\Lambda}_1 & \cdots & \mathbf{\Lambda}_N \end{bmatrix} \mathbf{\Delta}_\omega = \tilde{\mathbf{L}}, \quad \text{a triangular factor.} \tag{A.2}$$

Therefore using the $\mathrm{UDU}^T$ factorization of $\mathbf{R}_N^{-1}$, we have

$$\begin{bmatrix} \mathbf{\Lambda}_0 & \mathbf{\Lambda}_1 & \cdots & \mathbf{\Lambda}_N \end{bmatrix} \mathbf{\Delta}_\omega = \mathrm{UDU}^T \tilde{\mathbf{L}}, \tag{A.3}$$

The same exercise can be performed with any nested sequence of right-hand-side vectors; i.e., consider

$$\mathbf{R}_N \begin{bmatrix} \gamma_0(0) & \gamma_1(0) & \cdots & \gamma_N(0) \\ 0 & \gamma_1(1) & \cdots & \gamma_N(1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_N(N) \end{bmatrix} = \begin{bmatrix} f_0 & f_0 & f_0 & \cdots & f_0 \\ \times & f_1 & f_1 & \cdots & f_1 \\ \times & \times & f_2 & \cdots & f_2 \\ \vdots & \vdots & \vdots & & \vdots \\ \times & \times & \times & \cdots & f_N \end{bmatrix}. \tag{A.12}$$

In this case we multiply both sides of (A.12) by the matrix

$$\mathbf{\Delta}_{-1} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & \ddots & \\ & & & \ddots & -1 \\ & & & & 1 \end{bmatrix}, \tag{A.13}$$

and we obtain

$$\underset{\text{upper triangular}}{\mathbf{R}_N[\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \dots, \mathbf{\Gamma}_N]\mathbf{\Delta}_{-1}} = \underset{\text{lower triangular}}{\begin{bmatrix} f_0 & f_0 & f_0 & \cdots & f_0 \\ \times & f_1 & f_1 & \cdots & f_1 \\ \times & \times & f_2 & \cdots & f_2 \\ \vdots & \vdots & \vdots & & \vdots \\ \times & \times & \times & \cdots & f_N \end{bmatrix}} \mathbf{\Delta}_{-1}. \tag{A.14}$$

As before, we have therefore

$$[\mathbf{\Gamma}_0, \quad \mathbf{\Gamma}_1 - \mathbf{\Gamma}_0 \quad ,\dots, \quad \mathbf{\Gamma}_N - \mathbf{\Gamma}_{N-1}] = \mathrm{UDU}^T\tilde{L}, \tag{A.15}$$

which provides

$$\operatorname{diag}\{\gamma_i^2(i)d_i^{-1}\} = [\mathbf{\Gamma}_0, \mathbf{\Gamma}_1 - \mathbf{\Gamma}_0 \quad ,\dots, \quad \mathbf{\Gamma}_N - \mathbf{\Gamma}_{N-1}]^T \times \begin{bmatrix} f_0 & f_0 & f_0 & \cdots & f_0 \\ \times & f_1 & f_1 & \cdots & f_1 \\ \times & \times & f_2 & \cdots & f_2 \\ \vdots & \vdots & \vdots & & \vdots \\ \times & \times & \times & \cdots & f_N \end{bmatrix} \mathbf{\Delta}_{-1}. \tag{A.16}$$

This then implies that

$$\frac{\gamma_n^2(n)}{d_n} = \langle \Gamma_n, [f_0, f_1, \dots, f_n, \times, \times, \times] \rangle$$

$$- \langle \Gamma_{n-1}, [f_0, f_1, \dots, f_{n-1}, f_n, \times, \times, \times] \rangle. \qquad \text{(A.17)}$$

Defining a generalized mass sequence $\{M_F(n),\ n = 0, 1, 2, \dots, N\}$ via

$$M_F(n) = \begin{bmatrix} \gamma_n(0) & \cdots & \gamma_n(n) \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}, \qquad \text{(A.18)}$$

we obtain that the diagonal elements are expressed in terms of the generalized mass sequence as follows:

$$d_n = \frac{\gamma_n^2(n)}{M_F(n) - M_F(n-1)}. \qquad \text{(A.19)}$$

The above relations can be used to derive inversion algorithms, based on solutions of equations of the type (4.1).

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*Received 8 August 1984; revised 13 November 1984*