# On Holographic Transform Compression of Images

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**ABSTRACT:** Lossy transform compression of images is successful and widespread. The JPEG standard uses the discrete cosine transform on blocks of the image and a bit allocation process that takes advantage of the uneven energy distribution in the transform domain. For most images, 10:1 compression ratios can be achieved with no visible degradations. However, suppose that multiple versions of the compressed image exist in a distributed environment such as the internet, and several of them could be made available upon request. The classical approach would provide no improvement in the image quality if more than one version of the compressed image became available. In this paper, we propose a method, based on multiple description scalar quantization, that yields decompressed image quality that improves with the number of compressed versions available. © 2001 John Wiley & Sons, Inc. Int J Imaging Syst Technol, 11, 292–314, 2000

**Key words:** holographic representations; multiple descriptors

## I. INTRODUCTION

Bruckstein et al. (1998) introduced so-called holographic representations for still images. In this context, holographic means whole picture, and such representations enable the reconstruction of a complete image from portions of its representation with quality dependent on the size of the portion available. If we break the representation into several packets, the holographic property of the representation discussed in Bruckstein et al. (1998) enables progressive image recovery with the packets received in random orderings. This is in contrast to the traditional approaches for progressive coding of images for which the most important low frequency information is transmitted (and has to be received) first and subsequent data provide refinements on the information encoded previously. This paper tests ideas of combining multiple description scalar quantization with the bit allocation process of transform compression algorithms to achieve holographic compression. In particular, several compressed versions of the image are generated, which are all similar in quality. However, if more than one version is available to the decoder, image reconstructions will improve with the number of compressed versions available.

After transforming blocks of the image, we gather statistics about the variables in the transform domain (i.e., the discrete cosine transform [DCT] representation coefficients) and, as in classical transform coding, we are led to a bit allocation procedure. The gray levels of the image in an $n \times n$ ($8 \times 8$) block are mapped to $n^2$ (64) coefficients and their statistics over the image are calculated. Hence, we have coefficient vectors with known mean values and variances. We can then allocate a total given number $B$ of bits to the coefficients to minimize the quantization error induced, exactly as is done in classical transform compression. However, once the number of bits is assigned to each of the $n^2$ (64) coefficients, we designate for each coefficient a number $L$ of slightly different scalar quantizers (we choose $L = 8$) that will provide information about the coefficient that is mapped through it. Therefore, each of the block transform coefficients is assigned its multiple descriptor quantizer having $L$ channels. The multiple descriptor quantizer divides the range of the continuous transform variable (coefficient) into $L$ partitions. Each is assigned to a different channel, providing slightly different compressed versions of the image. If more than one compressed version is received by the decoder, a finer partition of the variable range is induced, leading to better estimates for the transform coefficients.

Therefore, when there are $B$ bits per channel quantization of the image, a certain quality for the reconstructed image (induced by the quantization errors) is obtained. If we get more than one channel, i.e., a total of $2B$, $3B$, . . . , $kB$, . . . , up to $LB$ bits per block in the image (which translates to a certain number of bits per pixel), we shall have image reconstruction that improves with increasing $k$ (provided the multiple description scalar quantizers are designed carefully).

The next section describes multiple description quantization and analyzes the errors incurred in quantizing uniform continuous random variables with multiple descriptions. The ideas are straightforward extensions of work in information theory on multiple description scalar quantization (Vaishampayan, 1993; where unfortunately "multiple" is always "2"). We also present multiple description transform coding for images and show experiments that demonstrate the holographic properties.

## II. PARTITIONING OF THE UNIT INTERVAL

If $n - 1$ points are to be distributed on the unit interval to form $n$ subintervals, the expected error when a uniform random variable $X$ over [0, 1] is quantized to take as values the midpoints of these subintervals is minimized when the subintervals are all the same length. When dealing with multiple channels, the question arises: Given $n - 1$ evenly spaced points on the unit interval and a positive integer $l$, how should the points be divided into $l$ sets so that the expected error of $X$ is minimized when it is quantized over each of the $l$ sets? If $E[\Delta X(S_i)]^2$ represents the expected squared error when $X$ is quantized over the subintervals formed by the points in $S_i$, then we seek to minimize $\max_i E[\Delta X(S_i)]^2$.

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First, considering the case $l = 1$, a simple calculation shows that if $n - 1$ points are to be distributed on the unit interval to form $n$ subintervals, the expected error when a uniform random variable $X$ is quantized to take as values the midpoints of these subintervals is minimized when the subintervals are all the same length. This error is

$$[E(\Delta X)^2]^{1/2} = \left[ \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} \left( x - \frac{2i+1}{2n} \right)^2 dx \right]^{1/2}$$

$$= \left[ \frac{2}{3} \sum_{i=0}^{n-1} \left( \frac{1}{2n} \right)^3 \right]^{1/2} = \frac{1}{2\sqrt{3}n}. \qquad (1)$$

(For comparison, the average error when placing $n - 1$ break points at random but uniformly in the intervals $[(i - 1)/(n - 1), i/(n - 1)]$ is larger than that given by Eq. (1), but is also asymptotic to $1/(2\sqrt{3}n)$ as $n \to \infty$. If we place the points totally at random within the interval [0, 1], and quantize $X$ using the ordered set of random points as the break points, then the asymptotic root mean square error behaves as $1/(\sqrt{2}n)$.)

We consider the situation where we partition $l(m - 1)$ evenly spaced points into $l$ sets of $m - 1$ points, so that each set by itself partitions the unit interval into $m$ subintervals, and $n$, the total number of subintervals, is $(m - 1)l + 1$. We will show that in this case, the set $\{1/n, 2/n, \ldots, (n - 1)/n\}$ should be divided into subsets $S_j = \{j/n, (j + l)/n, (j + 2l)/n, \ldots, [j + (m - 2)l]/n\}$, $j = 1, \ldots, l$.

Indeed, let the points in $S_j$ be denoted by $\{p_i^{(j)}\}$, where $p_1^{(j)} \le p_2^{(j)} \le \cdots \le p_{m-1}^{(j)}$, $j = 1, \ldots, l$. By straightforward algebra,

$$E[\Delta X(S_j)]^2 = \int_0^1 [x - Q(x)]^2\, dx$$

$$= \frac{1}{12} \left[ p_1^{(j)^3} + \sum_{i=1}^{m-2} (p_{i+1}^{(j)} - p_i^{(j)})^3 + (1 - p_{m-1}^{(j)})^3 \right],$$

where $x$ denotes the values $X$ can take and $Q(x)$ their quantized versions. Therefore, this problem is equivalent to finding the $l$ disjoint sets $S_j$ so that the maximum of

$$p_1^{(j)^3} + \sum_{i=1}^{m-2} (p_{i+1}^{(j)} - p_i^{(j)})^3 + (1 - p_{m-1}^{(j)})^3 \qquad (2)$$

over $j = 1, \ldots, l$ is minimized.

(If $S_j$ does equal $\{j/n, (j + l)/n, (j + 2l)/n, \ldots, [j + (m - 2)l]/n\}$, then the sum in Eq. (2) is

$$\frac{j^3 + (m-2)l^3 + (l - j + 1)^3}{[(m-1)l + 1]^3}.$$

For $1 \le j \le l$, this expression is largest when $j = 1$ or $j = l$, and there it takes on the value

$$\frac{(m-1)l^3 + 1}{[(m-1)l + 1]^3}.$$

We will show that this value cannot be improved.)

Without loss of generality, we can assume $1/n \in S_1$, so that $p_1^{(1)} = 1/n$. Now we try to choose $m - 2$ more points from the set $\{2/n, 3/n, \ldots, (n - 1)/n\}$ to be added to the set $S_1$ so that the expression in Eq. (2) is minimized. Define the differences

$$a_i^{(j)} = \left\{ \begin{array}{ll} p_1^{(j)} & \text{if } i = 0 \\ p_{i+1}^{(j)} - p_i^{(j)} & \text{if } 1 \le i \le m-2 \\ 1 - p_{m-1}^{(j)} & \text{if } i = m-1. \end{array} \right\}.$$

Then, the problem of minimizing $S_1$ is equivalent to

$$\text{Minimize } \sum_{i=1}^{m-1} a_i^{(1)^3}$$

$$\text{Subject to } \sum_{i=1}^{m-1} a_i^{(1)} = \frac{n-1}{n},\ a_i^{(1)} > 0.$$

This can be solved readily using Lagrange multipliers. The solution is $a_i^{(1)} = l/n$ for $i = 1, \ldots, m - 1$. Thus, the optimal set of points for $S_1$ given that it includes $p_1^{(1)} = 1/n$ is $\{1/n, (l + 1)/n, \ldots, [(m - 2)l + 1]/n\}$. This gives a sum of $[l^3(m - 1) + 1]/n^3 = [l^3(m - 1) + 1]/[l(m - 1) + 1]^3$ for Eq. (2).

Because the optimal solution for $S_1$ given that it contains $1/n = 1/[(m - 1)l + 1]$, disregarding the other sets, produces $[(m - 1)l^3 + 1]/[(m - 1)l + 1]^3$ for the value of Eq. (2), and because a partition of the points $\{i/n\}$ exists where this is the maximal error attained over all $S_j$s, the value $[(m - 1)l^3 + 1]/[(m - 1)l + 1]^3$ is the least possible maximum value of Eq. (2) over all possible partitions of the $n = (m - 1)l + 1$ evenly spaced points.

Next, we consider the error when more than one of the $S_j$s is received. Say $k$ out of a total possible $l$ signals are received, so that $1 \le k \le l$. For this situation, we have the following (Fig. 1).

**Proposition 1:** *Let $S_j = \{j/n, (j + l)/n, (j + 2l)/n, \ldots, [j + (m - 2)l]/n\}$, $j = 1, \ldots, l$, where $n = (m - 1)l + 1$. Let $k$ distinct integers from $\{1, \ldots, l\}$ be chosen randomly, and denote these by $a_j$, where $a_1 < a_2 < \cdots < a_k$. The points in $S_{a_1}, \ldots, S_{a_k}$ then divide the unit interval into $(m - 1)k + 1$ subintervals. A*

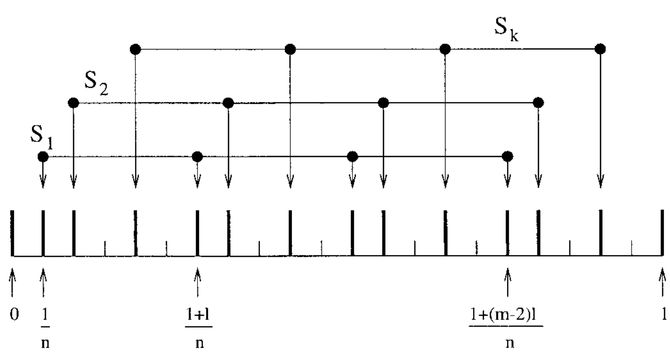


**Figure 1.** Partitioning of interval and quantization of signal. The channel $a_k$ quantizes the signal into the subintervals given by $S_k$. With more than one channel receiving the signal, the signal is quantized into the subintervals determined by all the $S_k$. These are the subintervals whose endpoints are the thick line segments. In this illustration, $k = 3$, $l = 5$, $m = 5$, $n = 21$, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$.

*random variable X, uniformly distributed on the unit interval, is quantized to $(m-1)k+1$ values by mapping its realizations into the midpoints of the subintervals to which they belong. The expected (root mean square) error incurred by this quantization is given by*

$$[E(\Delta X)^2]^{1/2} = [f(k, l, m)]^{1/2}$$

$$= \frac{\begin{array}{l}\{[(k-1)(m-1)+2](l+1) \\ \times [6l^2 - 6(k-2)l + (k-1)(k-6)] \\ + (m-2)(2l-k+1) \\ \times [12l^2 - 12(k-1)l + (k-1)(k-6)]\}^{1/2}\end{array}}{2\sqrt{3}\,\sqrt{(k+1)(k+2)(k+3)}n^{3/2}}. \quad (3)$$

The proof is given in the appendix.

When $k = l$, Eq. (3) reduces to $1/(2\sqrt{3}\,n)$, in accord with Eq. (1).

If $k$ and $l$ are fixed and $m \to \infty$, then the expected error (Eq. 3) approaches

$$\frac{\sqrt{6l^2 - 6kl + k^2 + 6l - 3k + 2}}{2\sqrt{3}\,\sqrt{(k+1)(k+2)lm}}.$$

If $k$ (and $l$) are large, this is approximately

$$\frac{\sqrt{6l^2 - 6kl + k^2}}{2\sqrt{3}\,klm}.$$

Furthermore, if $k$ is small compared with $l$, the error is approximately

$$\frac{1}{\sqrt{2}\,km}.$$

If we set $n = (m-1)k + 1$ in Eq. (1), we see that this error tends to $\sqrt{6}$ times the error in Eq. (1) as $m \to \infty$.

## III. HOLOGRAPHIC QUANTIZATION OF TRANSFORM COEFFICIENTS

In this section, we apply the multiple channel quantization method described in the previous section to block-transform coding of images. The block transform coefficients are random variables assumed to be described by their mean and variance. The bits allocated to represent them must be assigned to each coefficient according to a bit allocation scheme. The classical bit allocation scheme is based on reports by Huang and Schultheiss (1963) and Segall (1976).

The goal of bit allocation is to minimize the overall mean square error in the reconstructed image given a limited number of bits with which to represent the transformed image. One method (Huang and Schultheiss, 1963; Segall, 1976), which is described below, assumes that the total number of bits is fixed and seeks to minimize the total error by determining to which random variable subsequent bits should be assigned, until that number of bits has been assigned. Another approach is provided by Bruckstein (1987), who reported that the user has the option to strike a balance between the total quantization error and the number of bits used. In this method, which generalizes that where the total number of bits is absolutely fixed, positive weights are assigned to both the quantization error and number of bits used, and the bits are allocated to minimize the resulting sum.

In the present situation, a $2^N \times 2^N$ image is divided into $2^{2N-6}$ $8 \times 8$ blocks of pixels and the two-dimensional (2-D) discrete cosine transform is applied to these blocks. We can consider $Z_j$, $j = 1, \ldots, 64$ to be random variables with $2^{2N-6}$ sample values that are the entries of the $8 \times 8$ transformed blocks and let $\sigma_j^2$ be their variances. In order to be sure an a priori specified number $B$ of bits is used, the quantities given by

$$P_{j\beta} = \frac{3\sigma_j^2}{2^{2\beta+2}}, \quad j = 1, \ldots, 64, \quad \beta = 0, 1, \ldots,$$

are computed and arranged in decreasing order (Segall, 1976). The quantities $P_{j\beta}$ represent marginal returns. They measure the decrease in the total mean square error obtained by using an additional bit to represent the $j$th component, where the mean square error is estimated by a constant times $\sigma^2/(2^{2\beta})$. $\beta$ represents the numbers of bits used for a particular component and $2^\beta$ is the number of levels of quantization. Let the $B$th largest value of these $P_{j\beta}$ be $B_0$. The number of bits assigned to the $j$th component of the $8 \times 8$ array is the integer $b_j$, which satisfies $P_{j,b_j-1} \geq B_0 > P_{j,b_j}$. If $P_{j0} < B_0$ then $b_j = 0$ and if several of the $P_{j\beta}$ exactly equal $B_0$, any of them may be chosen so that the total number of bits is $B$. An example of such a bit allocation of 0.5 bits per pixel is shown in Figure 2. In this case, $B = 32$ and $B_0 = 31.90$. We show the part of the table where the entries are at least $B_0/4$.

If $b_j$ bits are assigned to the position $j$, then $Z_j$ may take on any of $m = 2^{b_j}$ values. Let the mean of $Z_j$ be $\mu_j$ and the variance be $\sigma_j$. We assume that the $Z_j$ are normally distributed. Prior to quantization, we transform conceptually the $Z_j$ into a random variable that is uniformly distributed. We then quantize this transformed variable. Although this approach is suboptimal, it works quite well for the low bit rates allocated to each transform coefficient in our examples. Define $N(z;\, \mu,\, \sigma)$ to be the Gaussian density

$$N(z;\, \mu,\, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(t-\mu)^2/2\sigma^2}$$

and $\Phi(z;\, \mu,\, \sigma)$ to be the distribution function

$$\Phi(z;\, \mu,\, \sigma) = \int_{-\infty}^{z} N(t;\, \mu,\, \sigma)dt.$$

Then the inverse function $\Phi^{-1}(w;\, \mu,\, \sigma)$ is defined for $0 < w < 1$. Let $a_1, \ldots, a_k$ be natural numbers that specify the quantization channels over which the image information is made available, i.e., the transformed versions of the image that are made available for the decoder, with $1 \leq a_1 < a_2 < \cdots < a_{k-1} < a_k \leq l$. Also define $n = (m-1)l + 1$. The unit interval is divided into $n$ subintervals defined by the break points $(a_h + il)/n$, $h = 1, \ldots, k$, $i = 0, 1, \ldots, (m-2)$. Each channel $a_h$ divides the unit interval into $m$ subintervals. The value of $Z_j$ is quantized in the following manner. $\Phi(z_j;\, \mu_j,\, \sigma_j)$ is computed, producing a number in (0, 1). The subinterval in which it falls is determined by computing

$$q_h = \left\lfloor \frac{n\Phi(z_j;\, \mu_j,\, \sigma_j) - a_h}{l} \right\rfloor$$

$P_{j\beta}$

| $j$ \ $\beta$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 1 | **34600** | **8650** | **2163** | **541** | **135** | **34** | 8 |
| 2 | **1059** | **265** | **66** | 17 | | | |
| 3 | **321** | **80** | 20 | | | | |
| 4 | **134** | **34** | 8 | | | | |
| 5 | **70** | 17 | | | | | |
| 6 | **59** | 15 | | | | | |
| 7 | **41** | 10 | | | | | |
| 8 | 24 | | | | | | |
| 9 | **969** | **242** | **61** | 15 | | | |
| 10 | **202** | **50** | 13 | | | | |
| 11 | **74** | 18 | | | | | |
| 12 | **36** | 9 | | | | | |
| 13 | 20 | | | | | | |
| 14 | 16 | | | | | | |
| 15 | 14 | | | | | | |
| 17 | **275** | **69** | 17 | | | | |
| 18 | **115** | 29 | | | | | |
| 19 | **53** | 13 | | | | | |
| 20 | 25 | | | | | | |
| 21 | 15 | | | | | | |
| 22 | 12 | | | | | | |
| 23 | 10 | | | | | | |
| 25 | **120** | 30 | | | | | |
| 26 | **59** | 15 | | | | | |
| 27 | **32** | 8 | | | | | |
| 28 | 17 | | | | | | |
| 29 | 12 | | | | | | |
| 30 | 9 | | | | | | |
| 33 | **56** | 14 | | | | | |
| 34 | **35** | 9 | | | | | |
| 35 | 24 | | | | | | |
| 36 | 12 | | | | | | |
| 37 | 8 | | | | | | |
| 41 | 28 | | | | | | |
| 42 | 23 | | | | | | |
| 43 | 16 | | | | | | |
| 49 | 15 | | | | | | |
| 50 | 13 | | | | | | |
| 51 | 12 | | | | | | |
| 57 | 8 | | | | | | |
| 58 | 11 | | | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 3 | 2 | 2 | 1 | 1 | 1 | 0 |
| 3 | 2 | 1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Figure 2.** Bit allocation array for the Goldhill image at 0.5 bpp. The number of bits assigned to the *j*th component is equal to the number of entries exceeding 31.9 (shown in boldface) in the *j*th row of the table.

for $h = 1, \ldots, k$, and selecting $q$ to be the largest of these $q_h$. The endpoints of the selected subinterval are generally either $(a_h + ql)/n$ and $(a_{h+1} + ql)/n$ or $(a_k + ql)/n$ and $[a_1 + (q+1)l]/n$ if $h = k$. Otherwise, $\Phi(z_j;\, \mu_j,\, \sigma_j)$ falls into one of the extreme subintervals $[0,\, a_1/n]$ or $[[a_k + (m-2)l]/n,\, 1]$. If the endpoints of this interval are denoted by $c_{i-1}$ and $c_i$, the quantized value of $Z_j$ is taken to be the expected value of $\Phi(z_j;\, \mu_j,\, \sigma_j)$ over the interval $[\Phi^{-1}(c_{i-1};\, \mu_j,\, \sigma_j),\, \Phi^{-1}(c_i;\, \mu_j,\, \sigma_j)]$. Explicitly, this value is

$$Q_{ji} = \mu_j + \frac{\sigma_j}{\sqrt{2\pi}} \frac{e^{-[\Phi^{-1}(c_{i-1};\mu_j,\sigma_j)-\mu_j]^2/2\sigma_j^2} - e^{-[\Phi^{-1}(c_i;\mu_j,\sigma_j)-\mu_j]^2/2\sigma_j^2}}{c_i - c_{i-1}}. \qquad (4)$$

Note that the $c_i$ depend on $l$, $m$, and the set $\{a_k\}$ of channels that receive the signal.

We simulated a transformed image being sent at a rate of 2, 4, and 8 bits per pixel (bpp) through eight channels, so that each channel carries 1/4, 1/2, and 1 bpp, respectively. In each of Figures 3, 4, and 5, (a) is the original image, and (b) is the reconstructed image using all eight channels. The subsequent pictures are the best and worst images obtained for successively fewer channels. As might be expected, the worst images (those with the smallest decibel values) are obtained when all the operating channels have consecutive $a_k$s.

The theoretical mean square error can be calculated as follows. If the quantization of the image $I(x, y)$ is represented as

$$\hat{I}(x, y) = \sum_{j=1}^{M} Q(z_j)\phi(x, y),$$

where the $Q(z_j)$ are the quantized transform coefficients and the $\phi(x, y)$ are an orthonormal basis, then the expected mean square error is

$$E(\hat{I} - I)^2 = \sum_{j=1}^{M} E[z_j - Q(z_j)]^2 = \sum_{j=1}^{M_1} E[z_j - Q(z_j)]^2 + \sum_{j=M_1+1}^{M} \sigma_j^2.$$

$\sigma_j^2$ is the variance of the $j$th component of the transform domain representation and $M_1$ is the number of components (out of the total of $M$, typically 64) assigned a nonzero number of bits. If $b_j$ bits are allocated in each channel to $z_j$ and $k$ channels out of a total of $l$ receive the signal, and if the $z_j$ variables were uniformly distributed, then

$$E[z_j - Q(z_j)]^2 = \left\{ \begin{array}{ll} 12\sigma_j^2 f(k, 1, 2^{b_j}) & \text{if } b_j > 0 \\ E(z_j - \overline{z}_j)^2 = \sigma_j^2 & \text{if } b_j = 0 \end{array} \right\},$$

where $f(k, l, m)$ is given by Eq. (3). The factor of 12 derives from the fact that the variance of a uniformly distributed random variable on the unit interval is 1/12. Hence

$$E(\hat{I} - I)^2 = \sum_{j=1}^{M_1} 12\sigma_j^2 f(k, l, 2^{b_j}) + \sum_{j=M_1+1}^{M} \sigma_j^2. \qquad (5)$$

However, the $z_j$s are not uniformly distributed; rather, we assume them to be Gaussian. Instead of $\sigma_j^2 f(k, l, 2^{b_j})$, we estimate the error in quantizing a Gaussian variable to approximately $k2^{b_j}$ levels as follows. Let the break points in the unit interval in a given partitioning to be labeled by $0 = c_0, c_1, \ldots, c_{N-1}, c_N = 1$, where $N = (m-1)k + 1$, and let $d_{ji} = \Phi^{-1}(c_i;\, \mu_j,\, \sigma_j)$. (Thus $d_{j0} = -\infty$ and $d_{jN} = +\infty$.) If $z_j \in [d_{j,i-1},\, d_{ji})$, define $Q(z_j)$ to be $Q_{ji}$, the quantized value of $Z_j$ for the corresponding subinterval. Also note that form Eq. (4),

$$\mu_j - Q_{ji} = \frac{\sigma_j}{\sqrt{2\pi}} \frac{e^{-(d_{ji}-\mu_j)^2/2\sigma_j^2} - e^{-(d_{i-1}-\mu_j)^2/2\sigma_j^2}}{c_i - c_{i-1}}.$$

Then the expected quantization error in this process is given by

$$E[z_j - Q(z_j)]^2 = \int_{-\infty}^{\infty} [z - Q(z)]^2 N(z;\, \mu_j,\, \sigma_j)\, dz$$


**Figure 3.** (a) Original test image of size 512 × 512. (b) Reconstructed image with eight channels, total of 2 bpp. (c), (d) Best and worst images with four channels, total of 1 bpp. (e), (f) Best and worst images with two channels, total of 0.5 bpp. (g), (h) Best and worst images with one channel, total of 0.25 bpp.

$$= \sum_{i=1}^{N} \int_{d_{j,i-1}}^{d_{ji}} (z - Q_{ji})^2 N(z;\, \mu_j,\, \sigma_j)\, dz$$

$$= \sum_{i=1}^{N} \int_{d_{j,i-1}}^{d_{ji}} [(z - \mu_j) + (\mu_j - Q_{ji})]^2 N(z;\, \mu_j,\, \sigma_j)\, dz$$

$$= \sum_{i=1}^{N} \int_{d_{j,i-1}}^{d_{ji}} (z - \mu_j)^2 N(z;\, \mu_j,\, \sigma_j)\, dz$$

$$+ 2 \sum_{i=1}^{N} (\mu_j - Q_{ji}) \int_{d_{j,i-1}}^{d_{ji}} (z - \mu_j) N(z;\, \mu_j,\, \sigma_j)\, dz$$

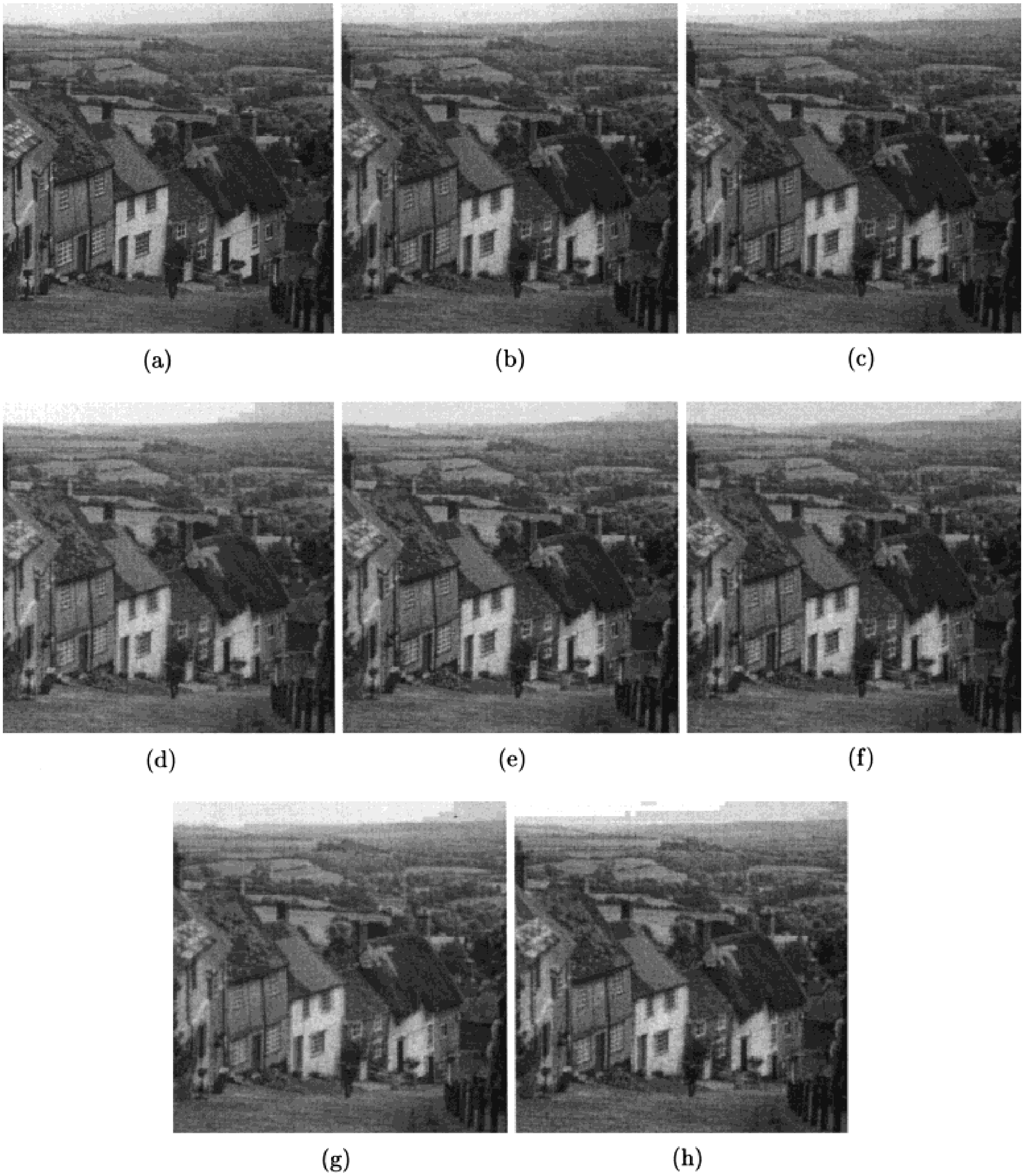

**Figure 4.** (a) Original test image of size 512 × 512. (b) Reconstructed image with eight channels, total of 4 bpp. (c), (d) Best and worst images with four channels, total of 2 bpp. (e), (f) Best and worst images with two channels, total of 1 bpp. (g), (h) Best and worst images with one channel, total of 0.5 bpp.

$$+ \sum_{i=1}^{N} (\mu_j - Q_{ji})^2 \int_{d_{j,i-1}}^{d_{ji}} N(z;\ \mu_j,\ \sigma_j)\ dz$$

$$= \sum_{i=1}^{N} \left\{ [-\sigma_j^2 (z - \mu_j) N(z;\ \mu_j,\ \sigma_j)]_{z=d_{j,i-1}}^{z=d_{ji}} \right.$$

$$+ \sigma_j^2 \int_{d_{j,i-1}}^{d_{ji}} N(z;\ \mu_j,\ \sigma_j)\ dz$$

$$+ 2 \sum_{i=1}^{N} [-\sigma_j^2 (\mu_j - Q_{ji}) N(z;\ \mu_j,\ \sigma_j)]_{z=d_{j,i-1}}^{z=d_{ji}}$$

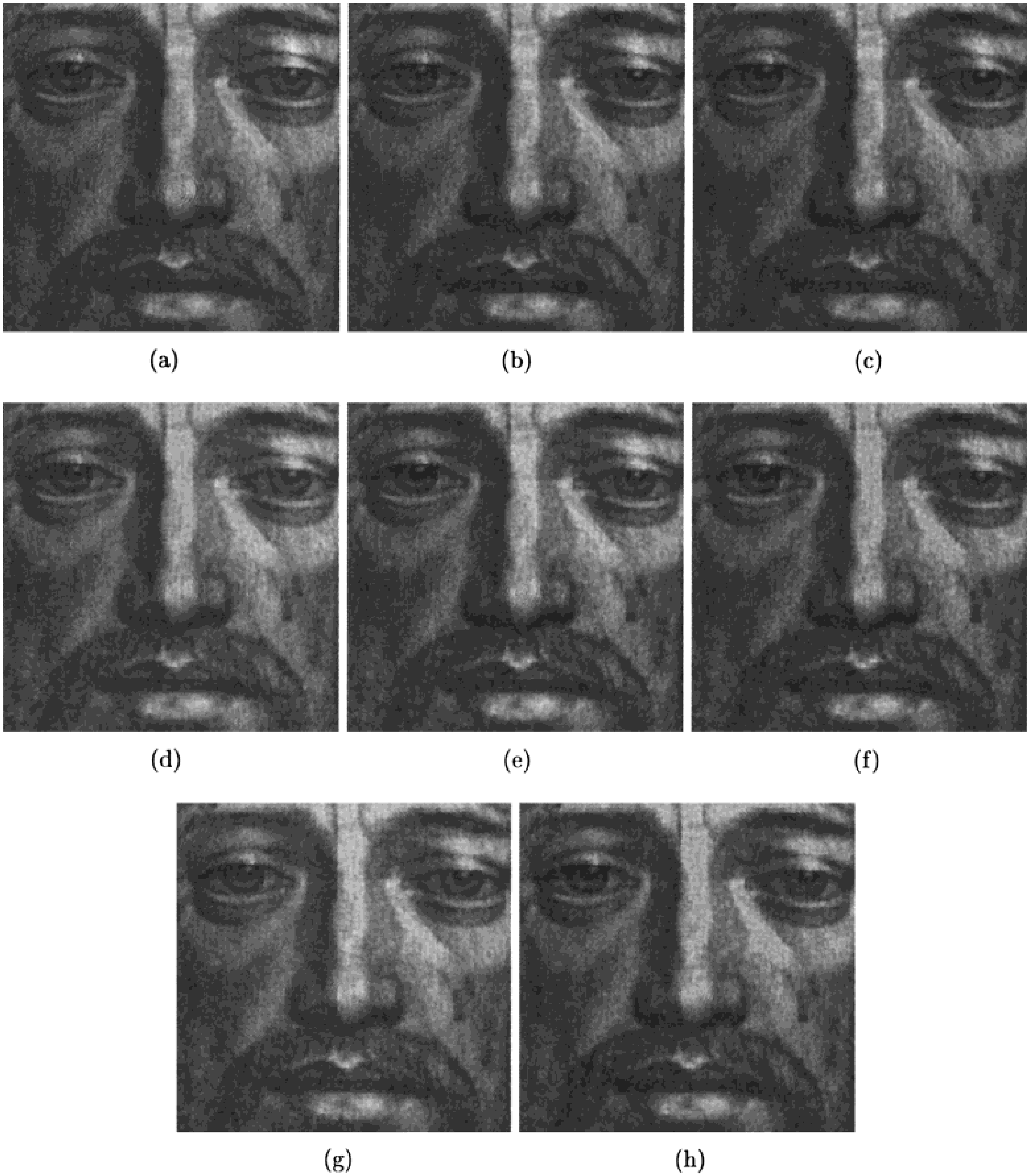


**Figure 5.** (a) Original test image of size 512 × 512. (b) Reconstructed image with eight channels, total of 8 bpp. (c), (d) Best and worst images with four channels, total of 4 bpp. (e), (f) Best and worst images with two channels, total of 2 bpp. (g), (h) Best and worst images with one channel, total of 1 bpp.

$$+ \sum_{i=1}^{N} \left[ (\mu_j - Q_{ji})^2 \int_{d_{j,i-1}}^{d_{ji}} N(z;\ \mu_j,\ \sigma_j)\ dz \right]$$

$$= -\sigma_j^2[\lim_{z\to\infty} (z - \mu_j)N(z;\ \mu_j,\ \sigma_j) - \lim_{z\to-\infty} (z - \mu_j)N(z;\ \mu_j,\ \sigma_j)]$$

$$+ \sigma_j^2 \int_{-\infty}^{\infty} N(z;\ \mu_j,\ \sigma_j)\ dz$$

$$- 2\sigma_j^2 \sum_{i=1}^{N} (\mu_j - Q_{ji})[N(d_{ji};\ \mu_j,\ \sigma_j) - N(d_{j,i-1};\ \mu_j,\ \sigma_j)]$$

$$+\sum_{i=1}^{N}(\mu_j - Q_{ji})^2[\Phi(d_{ji};\,\mu_j,\,\sigma_j) - \Phi(d_{j,i-1};\,\mu_j,\,\sigma_j)]$$

$$= -\sigma_j^2(0-0) + \sigma_j^2$$

$$-2\sigma_j^2\sum_{i=1}^{N}\left[\frac{\sigma_j}{\sqrt{2\pi}}\frac{e^{-(d_{ji}-\mu_j)^2/2\sigma_j^2} - e^{-(d_{j,i-1}-\mu_j)^2/2\sigma_j^2}}{c_i - c_{i-1}}\right]$$

$$\times\left[\frac{1}{\sqrt{2\pi}\,\sigma_j}\left(e^{-(d_{ji}-\mu_j)^2/2\sigma_j^2} - e^{-(d_{j,i-1}-\mu_j)^2/2\sigma_j^2}\right)\right]$$

$$+\sum_{i=1}^{N}\frac{\sigma_j^2}{2\pi}\left[\frac{e^{-(d_{ji}-\mu_j)^2/2\sigma_j^2} - e^{-(d_{j,i-1}-\mu_j)^2/2\sigma_j^2}}{c_i - c_{i-1}}\right]^2$$

$$\times(c_i - c_{i-1})$$

$$= \sigma_j^2\left\{1 - \frac{1}{2\pi}\sum_{i=1}^{N}\frac{[e^{-(d_{ji}-\mu_j)^2/2\sigma_j^2} - e^{-(d_{j,i-1}-\mu_j)^2/2\sigma_j^2}]^2}{c_i - c_{i-1}}\right\}.$$

Using the identity

$$d_{ji} = \Phi^{-1}(c_i;\,\mu_j,\,\sigma_j) = \mu_j + \sigma_j\Phi^{-1}(c_i;\,0,\,1),$$

we have

$$\frac{d_{ji}-\mu_j}{\sigma_j} = \Phi^{-1}(c_i;\,0,\,1),$$

and we can write the above equation as

$$E[z_j - Q(z_j)]^2 = \sigma_j^2 g(\{a_h\}_{h=1}^k,\,l,\,m), \qquad (6)$$

where

$$g(\{a_h\}_{h=1}^k,\,l,m) = \left\{1 - \frac{1}{2\pi}\sum_{i=1}^{N}\frac{[e^{-[\Phi^{-1}(c_i;0,1)]^2/2} - e^{-[\Phi^{-1}(c_{i-1};0,1)]^2/2}]^2}{c_i - c_{i-1}}\right\}.$$

$N = (m-1)k + 1$, and the $c_i$ are given by $c_0 = 0$, $c_N = 1$, and $c_{jk+h} = (jl + a_h)/n$, $j = 0, 1, \ldots, m-2$, $h = 1, \ldots, k$.

The values $g(\{a_h\}_{h=1}^k, l, 2^b)$ needed for the computation of the expression (Eq. 6) may be taken from a table. This table, a portion of which is shown as Table I, will have entries for every possible combination of channels receiving the signal and for every possible positive number of bits used to quantize a random variable, typically from 1 to 8. This table may be condensed by observing that due to symmetry, we have

$$g(\{a_h\}_{h=1}^k,\,l,\,2^b) = g(\{l+1-a_h\}_{h=1}^k,\,l,\,2^b).$$

Part of this table, for $l = 8$, is shown in Table I.

Figure 6 shows how the signal-to-noise ratio (SNR) increases with the number of channels carrying the signal. For a $2^N \times 2^N$ original image $I$ and a transformed image $\hat{I}$, the SNR is computed as

$$\mathrm{SNR}(I,\hat{I}) = -10\log_{10}\left(\frac{1}{2^{4N}}\left\{\sum_{i=1}^{2^N}\sum_{j=1}^{2^N}\left[\frac{\hat{I}(i,j) - I(i,j)}{I_{\max}}\right]^2\right\}\right).$$

(The individual value $I(i, j)$ is the gray-level value at pixel $(i, j)$, on a scale of 0 to $I_{\max}$, typically 255.) To match this definition, the expected error for uniformly distributed random variables in Eq. (5), and for Gaussian distributed random variables in Eq. (6), must be divided by $MI_{\max}^2$, which results in

$$E(SNR_{\mathrm{unif}}) = -10\log_{10}\left\{\frac{1}{MI_{\max}^2}\left[\sum_{j=1}^{M_1}12\sigma_j^2 f(k,\,l,\,2^{b_j}) + \sum_{j=M_1+1}^{M}\sigma_j^2\right]\right\} \qquad (7)$$

and

$$E(SNR_{\mathrm{Gaus}}) = -10\log_{10}\left\{\frac{1}{MI_{\max}^2}\left[\sum_{j=1}^{M_1}\sigma_j^2 g(\{a_h\}_{h=1}^k,\,l,\,2^{b_j}) + \sum_{j=M_1+1}^{M}\sigma_j^2\right]\right\}, \qquad (8)$$

**Table I.** Portion of table of values used to compute expected quantization error of Gaussian random variables.

| | $g(\{a_h\}_{h=1}^k,\,l,\,2^b)$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\{a_h\}$ $b$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| {1} or {8} | 0.6368 | 0.1548 | 0.0508 | 0.0188 | 0.0075 | 0.0031 | 0.0013 | 0.0006 |
| {2} or {7} | 0.4869 | 0.1348 | 0.0453 | 0.0169 | 0.0067 | 0.0028 | 0.0012 | 0.0005 |
| {3} or {6} | 0.4051 | 0.1240 | 0.0424 | 0.0159 | 0.0064 | 0.0026 | 0.0011 | 0.0005 |
| {4} or {5} | 0.3679 | 0.1190 | 0.0411 | 0.0155 | 0.0062 | 0.0026 | 0.0011 | 0.0005 |
| {1, 2} or {7, 8} | 0.4574 | 0.1128 | 0.0373 | 0.0139 | 0.0056 | 0.0023 | 0.0010 | 0.0004 |
| {1, 2, 3} | 0.3341 | 0.0832 | 0.0278 | 0.0105 | 0.0042 | 0.0018 | 0.0008 | 0.0003 |
| {1, 2, 3, 4} | 0.2426 | 0.0609 | 0.0206 | 0.0078 | 0.0032 | 0.0013 | 0.0006 | 0.0003 |
| {1, 2, 3, 4, 5} | 0.1723 | 0.0437 | 0.0149 | 0.0058 | 0.0024 | 0.0010 | 0.0004 | 0.0002 |
| {1, 2, 3, 4, 5, 6} | 0.1180 | 0.0303 | 0.0105 | 0.0041 | 0.0017 | 0.0007 | 0.0003 | 0.0001 |
| {1, 2, 3, 4, 5, 6, 7} | 0.0766 | 0.0201 | 0.0071 | 0.0028 | 0.0012 | 0.0005 | 0.0002 | 0.0001 |
| {1, 2, 3, 4, 5, 6, 7, 8} | 0.0470 | 0.0126 | 0.0045 | 0.0018 | 0.0008 | 0.0003 | 0.0002 | 0.0001 |

This table is for $l = 8$ total possible channels. The full table has rows for each nonempty combination of channels.

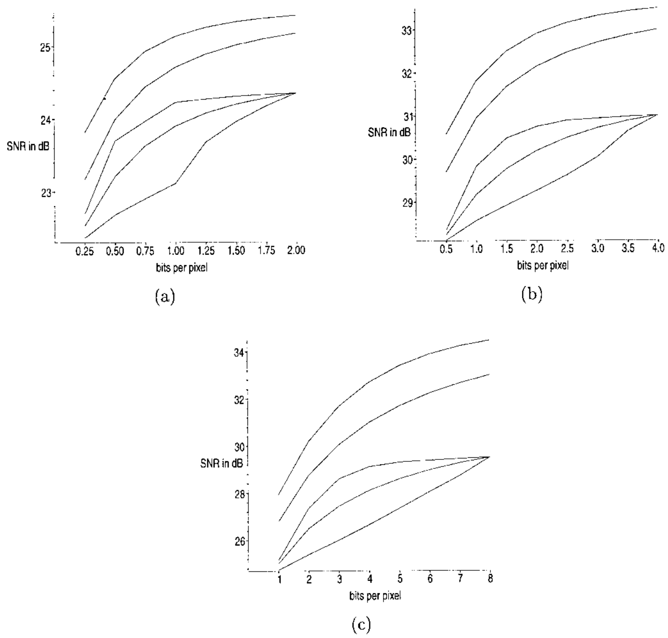


**Figure 6.** SNR values for the (a) Barbara image, (b) Goldhill image, and (c) scanned engraving as a function of the number of channels carrying the signal. From top to bottom, the graphs are the expected SNR values according to Eqs. (7) and (8) and the maximum, average, and minimum SNR values among all possible combinations of channels. Each of eight channels carries 1/4, 1/2, and 1 bpp in (a), (b), and (c), respectively, for totals of 2, 4, and 8 bpp if all channels are receiving the signal.

respectively. Included in Figure 6 are the expected SNR values according to Eqs. (7) and (8). These values differ from the observed SNR values because the actual distribution of the signals is not particularly close to being uniform or Gaussian, but they do serve as a useful approximation. In all cases, the Gaussian assumption was closer to reality than the uniform assumption.

The accuracy of these expressions was tested by creating two images whose $Z_j$ are uniformly and Gaussian distributed, with the same means and variances of the Barbara image. Thus, if the mean and variance of a particular $Z_j$ for the Barbara images were $\mu_j$ and $\sigma_j$, the $Z_j$ in the first new image are sampled uniformly from the interval $[\mu_j - \sqrt{3}\,\sigma_j, \mu_j + \sqrt{3}\,\sigma_j]$. Then, the particular realization of $Z_j$ are quantized as follows. Let $\Psi(z; \mu, \sigma)$ be the uniform distribution function

$$\Psi(z; \mu, \sigma) = \left\{ \begin{array}{cc} 0 & z \le \mu - \sqrt{3}\,\sigma \\ \dfrac{z - \mu - \sqrt{3}\,\sigma}{2\sqrt{3}\,\sigma} & \mu - \sqrt{3}\,\sigma \le z \le \mu + \sqrt{3}\,\sigma \\ 1 & z \ge \mu + \sqrt{3}\,\sigma \end{array} \right\}.$$

If $\Psi(z_j; \mu_j, \sigma_j)$ falls into the interval $[c_{i-1}, c_i]$, $z_j$ is assigned the average value of $\Psi(z_j; \mu_j, \sigma_j)$ over the interval $[\Psi^{-1}(c_{i-1}; \mu_j, \sigma_j), \Psi^{-1}(c_i; \mu_j, \sigma_j)]$, which is $\mu_j + \sqrt{3}\,\sigma_j(c_{i-1} + c_i - 1)$. Table II shows the observed and expected SNR for the image with the uniformly distributed $Z_j$ for $k = 1, \ldots, 8$ channels. An analogous procedure is performed on the $Z_j$ in the second new image, where these values are taken from the Gaussian distribution with

**Table II.** Observed average and expected SNR (from Eq. 7) for the image with uniformly distributed $Z_j$ with the same means and variances as the Barbara image.

| $k$ | Observed Average SNR | Expected SNR |
|---|---|---|
| 1 | 23.87 | 23.86 |
| 2 | 24.61 | 24.60 |
| 3 | 24.98 | 24.97 |
| 4 | 25.18 | 25.18 |
| 5 | 25.30 | 25.30 |
| 6 | 25.38 | 25.38 |
| 7 | 25.43 | 25.43 |
| 8 | 25.46 | 25.46 |

The average is taken over all combinations of $k$ channels selected from a total of eight.

mean $\mu_j$ and variance $\sigma_j^2$. Here the $Z_j$ are quantized according to the formula (4). Table III shows the observed and expected SNR for the image with the Gaussian distributed $Z_j$ for $k = 1, \ldots, 8$ channels.

The agreement between the expected and actual errors is good for images with $Z_j$ distributed either uniformly or Gaussian. For actual images, the distribution of the $Z_1$, the component in the upper left corner of the $8 \times 8$ arrays, resembles neither a uniform nor a Gaussian distribution. However, the distributions of the other $Z_j$ have some resemblance to Gaussian distributions. Figure 7 shows histograms of the 4096 values of $Z_1$, the transformed coefficients for the upper left corner of each of the $8 \times 8$ blocks, and of $Z_2$, the component to the right of $Z_1$, for the Barbara image. The points plotted represent the number of values that fall in the interval $[\mu_1 + (n - 0.1)\sigma_1, \mu_1 + (n + 0.1)\sigma_1)$, where $n$ is the number listed on the horizontal axis. For example, 364 of the $Z_1$ are in the interval $[\mu_1 - 0.9\sigma_1, \mu_1 + 1.1\sigma_1)$. Distributions for the $Z_j$ nearer the bottom right corner of the $8 \times 8$ arrays become flatter and more spread out. Because the $Z_1$ term is the most dominant one in terms of contributing to the overall error, attempts to estimate the average error by assuming an underlying uniform or Gaussian distribution are somewhat inaccurate, although the Gaussian assumption is markedly better due to the distributions of the less dominant terms.

Another measurement of interest is the comparison of these quantized images with images obtained as follows. Given a bit allocation (Fig. 2), a transformed pixel is assigned its exact value if it is allocated at least one bit and the value obtained by the above process (the mean of $Z_j$ for position $j$) if it is allocated zero bits. This represents a best possible image given a bit assignment that deletes certain transform coefficients. Figure 8 is similar to Figure 6, except

**Table III.** Observed average and expected SNR (from Eq. 8) for the image with Gaussian distributed $Z_j$ with the same means and variances as the Barbara image.

| $k$ | Observed Average SNR | Expected SNR |
|---|---|---|
| 1 | 23.45 | 23.31 |
| 2 | 24.25 | 24.13 |
| 3 | 24.68 | 24.57 |
| 4 | 24.94 | 24.84 |
| 5 | 25.11 | 25.02 |
| 6 | 25.22 | 25.15 |
| 7 | 25.31 | 25.24 |
| 8 | 25.37 | 25.30 |

The average is taken over all combinations of $k$ channels selected from a total of eight.

that our quantized images are being compared to these best possible images instead of the original images. Consequently, the SNR values are higher. The differences decrease as the number of pixels carried by each channel increases as more positions are allocated a positive number of bits.

A direct comparison of our implementation, which is a JPEG-style transform compression, to the standard JPEG is not meaningful. Our implementation does not apply the clever tricks that the JPEG standard uses in order to achieve as deep a compression as possible, i.e., thresholding, separate DC encoding, and efficient zero runlength encoding, followed by further string compression. We propose the use of standard JPEG with shifted quantizers, hence the performance of the compression with a single data set will be identical to JPEG. However, for analysis and experimental verification of this idea, we only implemented a transform coding of JPEG style in order to be able to concentrate on, and single out, the effect of using staggered quantizers.

Therefore, there is no need to compare our compression results with the classical JPEG performance. The analysis exhibits clearly the benefits of using holographic, multiple-description quantizers in conjunction with any compression method that uses transforms, followed by quantization. It also predicts the expected benefits of classical JPEG when used holographically.

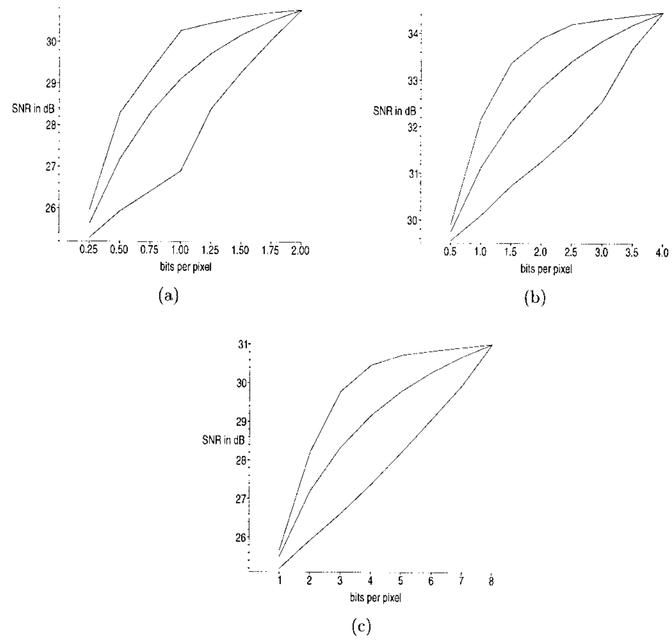


**Figure 8.** SNR values for the (a) Barbara image, (b) Goldhill image, and (c) scanned engraving compared with the image obtained by zeroing out the transformed pixels that are assigned no bits. From top to bottom, the graphs are the maximum, average, and minimum SNR values among all possible combinations of channels.

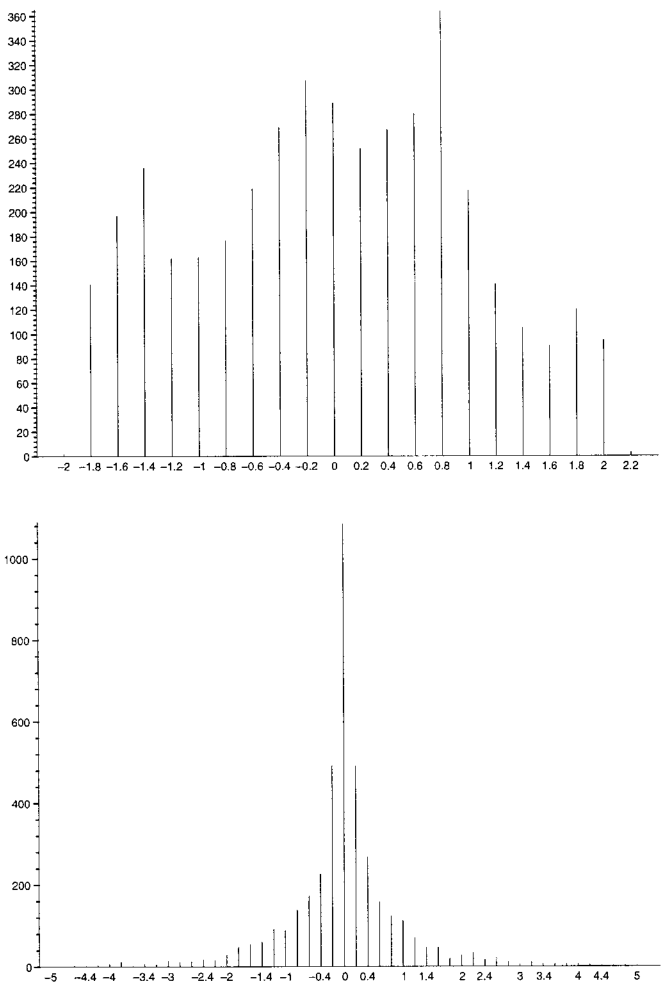

**Figure 7.** Histograms of the $Z_1$ (top) and $Z_2$ (bottom) values for the Barbara image. The horizontal axis represents the number of standard deviations away from the mean and is discretized to intervals of length $0.2\sigma_1$. The points plotted above the number $n$ on the horizontal axis are the number of the $Z_i$ that lie in the interval $[\mu_1 + (n - 0.1)\sigma_1, \mu_1 + (n + 0.1)\sigma_1]$.

## IV. RANDOMIZED HOLOGRAPHIC QUANTIZATION

If the number of possible available channels is not known beforehand, a viable quantization scheme can be obtained by dividing the unit interval into random subintervals. Specifically, for a channel $a_h$, $m - 1$ points are placed at $\{1/(2m - 2) + \epsilon_h, 3/(2m - 2) + \epsilon_h, \ldots, (2m - 3)/(2m - 2) + \epsilon_h\}$, where $\epsilon_h$ is chosen randomly and uniformly from $[-1/(2m - 2), 1/(2m - 2)]$.

The number of bits allocated to each coefficient of the transform domain representation is the same as in the previous section. Also, the quantization of an observation $Z_j$ is performed in a similar manner. If the signal is received by $k$ channels, each with an associated random value $\epsilon_h$, $h = 1, \ldots, k$, we order the channels so that $\epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_k$. We compute

$$q = \lfloor (m - 1)\Phi(z_j; \mu, \sigma) \rfloor.$$

Next, we find the largest $\epsilon_h$ satisfying

$$\epsilon_h < \Phi(z_j; \mu, \sigma) - \frac{2q + 1}{2m - 2},$$

in which case the endpoints of the selected subinterval are generally $(2q + 1)/(2m - 2) + \epsilon_h$ and $(2q + 1)/(2m - 2) + \epsilon_{h+1}$, or $(2q + 1)/(2m - 2) + \epsilon_k$ and $(2q + 3)/(2m - 2) + \epsilon_1$ if $h = k$. If $\epsilon_1 \geq \Phi(z_j; \mu, \sigma) - (2q + 1)/(2m - 2)$, then the endpoints are generally $(2q - 1)/(2m - 2) + \epsilon_k$ and $(2q + 1)/(2m - 2) +$

$\epsilon_1$. Otherwise, $\Phi(z_j; \mu, \sigma)$ falls into one of the extreme subintervals $[0, 1/(2m - 2) + \epsilon_1]$ or $[(2m - 3)/(2m - 2) + \epsilon_k, 1]$. The partitioning is shown in Figure 9.

We repeated the four examples in the previous section, where each of eight channels carries 1/4, 1/2, or 1 bpp. One thousand trials were run with $k$ channels receiving the signal for $k = 1, \ldots, 8$. The results are shown in Figures 10, 11, and 12. There is a greater variability in the results, with both the minimum and maximum SNR values being more extreme than for the fixed channels and the average value is slightly greater than before, particularly for the intermediate cases of two to seven channels.

To calculate the expected performance in terms of root mean square error in randomized holographic transform coding, we start by partitioning the unit interval into $n$ subintervals by choosing $n - 1$ random break points between 0 and 1. We have the following result:

**Proposition 2:** *Suppose $n - 1$ random points are i. i. d. (independent and identically distributed) uniformly on the unit interval. Let the ordered set of these points be denoted by $\{\mathbf{p}_i\} = \{p_1, \ldots, p_{n-1}\}$ where $0 \le p_1 \le p_2 \le \cdots \le p_{n-1} \le 1$. These points define n subintervals $[0, p_1)$, $[p_1, p_2)$, $\ldots$, $[p_{n-2}, p_{n-1})$, $[p_{n-1}, 1]$. A random variable X, also uniformly distributed on the unit interval, will be quantized to n values by mapping its realizations n into $Q(x)$, the midpoints of the subintervals to which they belong. The expected (root mean square) error incurred by this quantization is given by*

$$\frac{1}{\sqrt{2(n+1)(n+2)}}. \tag{9}$$

The proof is given in the appendix.

We now consider the situation where we want to partition the unit interval into $k$ sets of $m - 1$ points, where we assume $m$ is known and $k$ is an arbitrary positive integer (Fig. 9). We have this result:

**Proposition 3:** *For a given integer $m > 1$, define $s_i = (2i - 1)/[2(m - 1)]$, $i = 1, \ldots, m - 1$. Let $\epsilon_1, \ldots, \epsilon_k$ be chosen randomly and uniformly from the interval $(-1/[2(m - 1)], 1/[2(m - 1)])$, and order them so that $\epsilon_1 \le \epsilon_2 \le \ldots \le \epsilon_k$. Then the points $s_i + \epsilon_j$ divide the unit interval into $k(m - 1) + 1$ subintervals. A random variable X, uniformly distributed on the unit interval, is quantized to $k(m - 1) + 1$ values by mapping its realizations into the midpoints of the subintervals to which they belong. The expected (root mean square) error incurred by this quantization is given by*

$$[E(\Delta X)^2]^{1/2} = [g(k, m)]^{1/2} = \frac{\sqrt{(k+3)(m-1) - 2}}{\sqrt{2(k+1)(k+2)(k+3)}(m-1)^{3/2}}. \tag{10}$$

The proof is given in the appendix.

We observe that the number of subintervals in Proposition 2 is $n$, whereas it is $k(m - 1) + 1$ in Proposition 3. If we replace the quantity $n$ in Propositions 2 by $k(m - 1) + 1$, we find that the root mean square error in Eq. (9) is

$$[E(\Delta X)^2]^{1/2} = \frac{1}{\sqrt{2[k(m-1)+2][k(m-1)+3]}}. \tag{11}$$

As $k$ is held fixed and $m \to \infty$, Eqs. (10) and (11) are asymptotic to

$$\frac{1}{\sqrt{2(k+1)(k+2)}(m-1)}$$

and

$$\frac{1}{\sqrt{2}k(m-1)},$$

respectively. Thus, the latter gives a smaller expected error, although the difference between the two becomes negligible for large values of $k$. More precisely, the randomized method of Proposition 3 produces average errors that are $k/\sqrt{(k+1)(k+2)}$ times as large as the optimal partition for arbitrary $k \ge 2$ and large $m$. As $k$ grows without bound, this ratio of errors increases to 1.

Also, for fixed $k$, the error given by Eq. (10) is asymptotic to $1/[\sqrt{2(k+1)(k+2)}\,m]$ as $m \to \infty$, or equivalently is asymptotic to $k/[\sqrt{2(k+1)(k+2)}n]$ as $n \to \infty$. The method of Proposition 3 produces average errors that are $\sqrt{6}\ k/\sqrt{(k+1)(k+2)}$ times as large as the optimal partition for arbitrary $k$ and large $n$ given by Eq. (1). As $k$ grows without bound, this ratio of errors increases toward $\sqrt{6}$, thereby providing an upper bound on this ratio.

The theoretical mean square error can be calculated similarly to that in the previous section. In this case, we have

$$E(\hat{I} - I)^2 = \sum_{j=1}^{M_1} 12\sigma_j^2 g(k, 2^{b_j}) + \sum_{j=M_1+1}^{M} \sigma_j^2,$$

where $g(k, m)$ is given by Eq. (10) and

$$E(SNR_{\mathrm{unif}}) = -10\log_{10}\left\{\frac{1}{MI_{\max}^2}\left[\sum_{j=1}^{M_1} 12\sigma_j^2 g(k, 2^{b_j}) + \sum_{j=M_1+1}^{M} \sigma_j^2\right]\right\}. \tag{12}$$

Figure 13 shows how the SNR value increases with the number of channels carrying the signal. Included in these figures are the expected SNR values according to Eq. (12). As before, these values are greater that the observed values because the actual distributions of

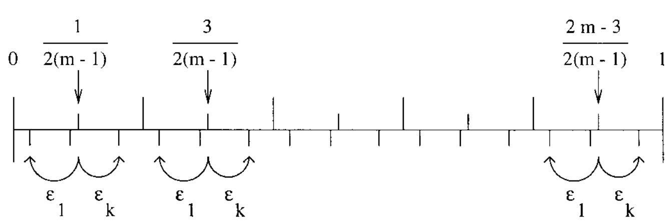


**Figure 9.** Randomized partitioning of interval and quantization of signal. The $k$th channel quantizes the signal into the subintervals given by $\{(2i - 1)/(2m - 2) + \epsilon_k;\ i = 1, \ldots, m - 2\}$. With more than one channel receiving the signal, the signal is quantized into the subintervals determined by all of these break points; these are the subintervals whose endpoints are the short line segments. In this illustration, $k = 3$, $m = 6$, $n = 16$, $\epsilon_1 = -3/40$, $\epsilon_2 = -1/80$, and $\epsilon_3 = 1/16$.


**Figure 10.** (a), (b) Best and worst images with randomized quantization and eight channels, total of 2 bpp. (c), (d) Best and worst images with four channels, total of 1 bpp. (e), (f) Best and worst images with two channels, total of 0.5 bpp. (g), (h) Best and worst images with one channel, total of 0.25 bpp.

the signals are not uniform. A similar expected error assuming a Gaussian distribution for the transform coefficients leads to very unwieldy expressions and will not be given here.

We can compare these quantized images to the best possible images of the previous section. The results shown in Figure 14 are similar to the results shown in Figure 13, with the quantized images now being compared with the best possible images instead of with the original images.

## V. CONCLUSIONS

We presented some new techniques for multiple descriptor quantization. Several slightly different compressed versions of an image

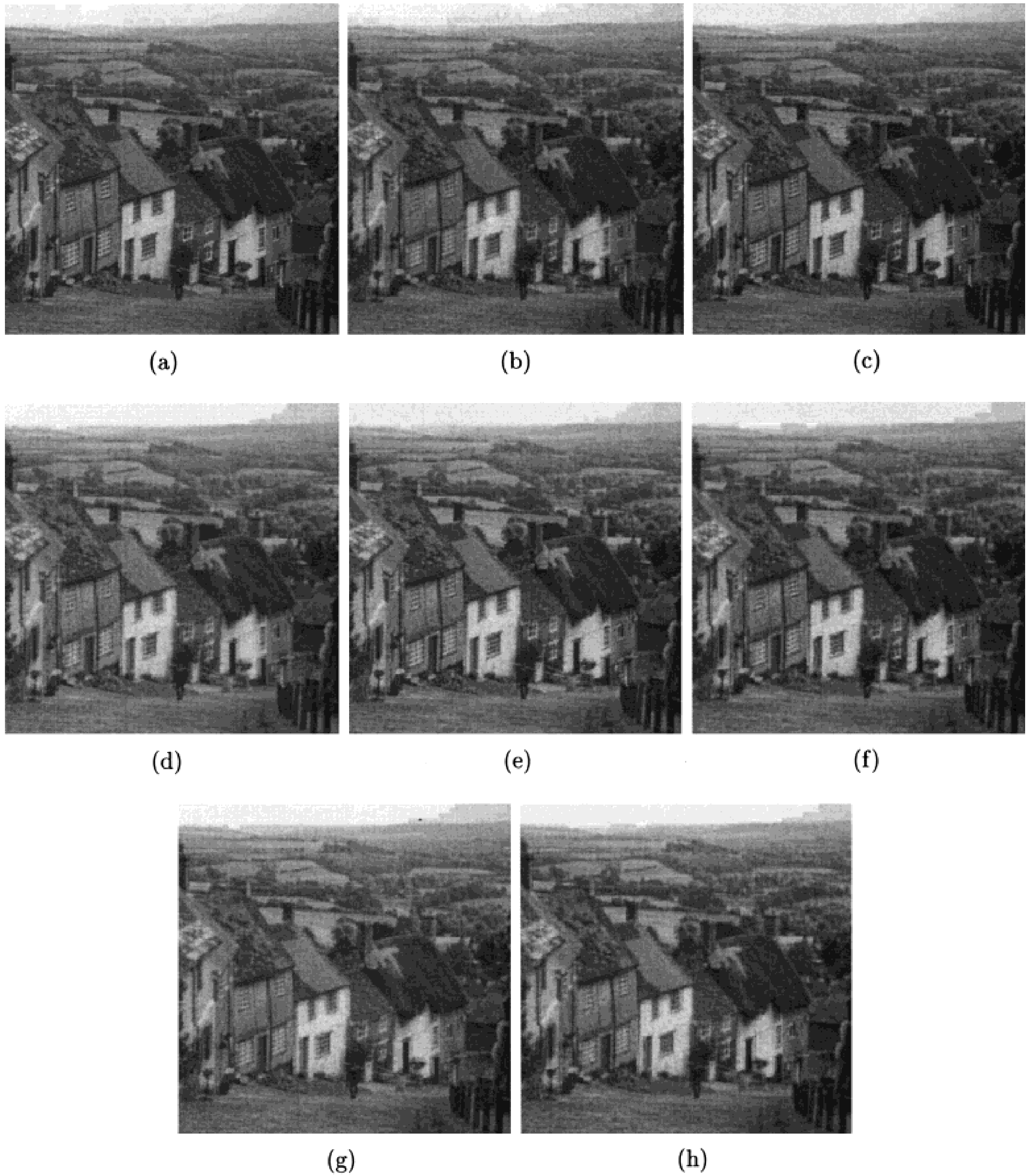


**Figure 11.** (a), (b) Best and worst images with randomized quantization and eight channels, total of 4 bpp. (c), (d) Best and worst images with four channels, total of 2 bpp. (e), (f) Best and worst images with two channels, total of 1 bpp. (g), (h) Best and worst images with one channel, total of 0.5 bpp.

are generated. As the number of these compressed versions that are transmitted increases, the quality of the reconstructed image improves. In particular, the improvement continues past the receipt of two versions.

In one setup, the unit interval is divided into subintervals of equal length and the quantizers are chosen to be staggered sets of break points. In another, the break points are chosen either randomly from the entire unit interval or from within equal-sized subintervals. In either case, the limiting behavior of these randomized quantizations is the same as the number of channels receiving the signal grows large. The expected root mean square error of these tends to be $\sqrt{6}$ times as large as that of the evenly staggered sets of break points.

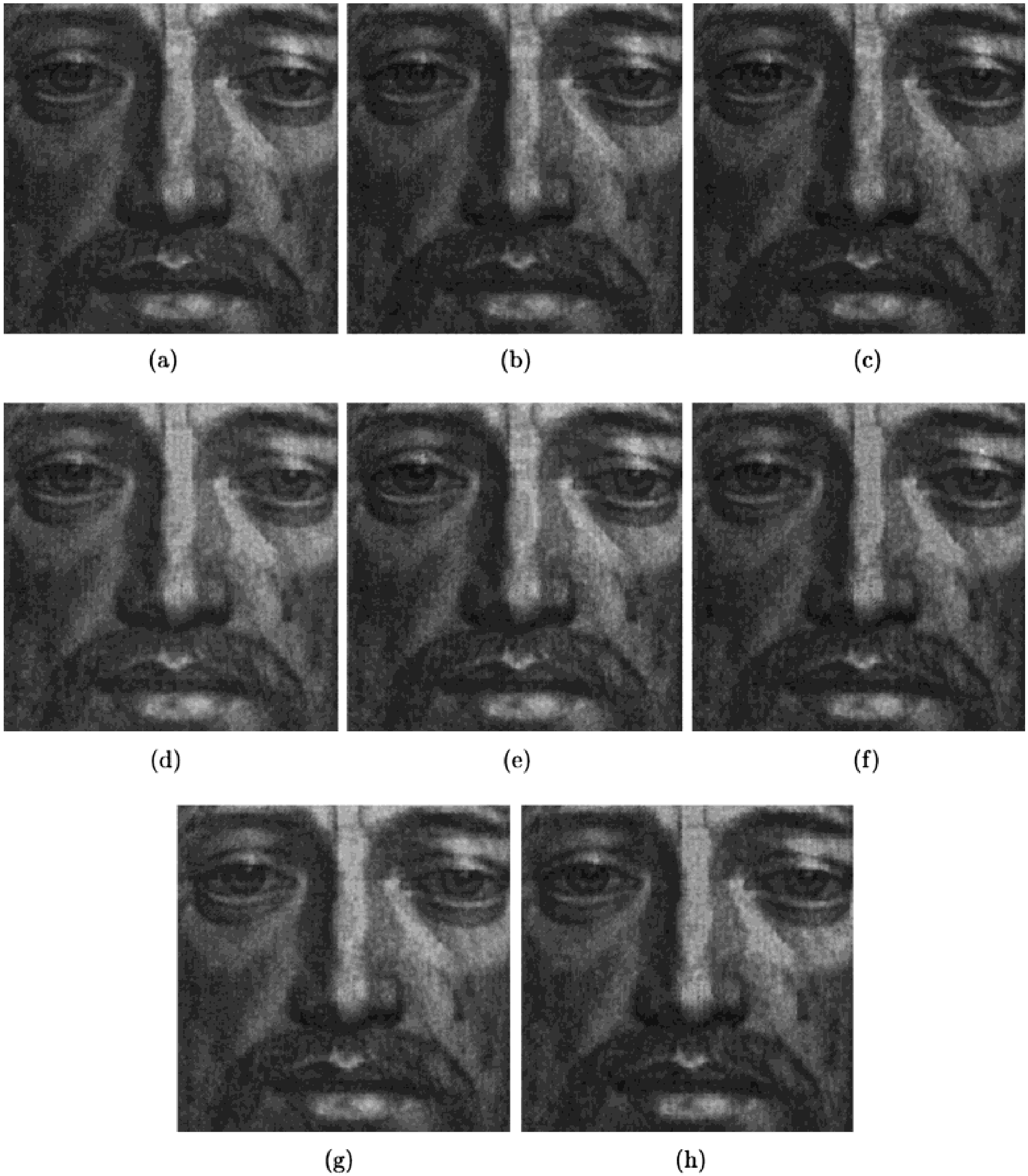


**Figure 12.** (a), (b) Best and worst images with randomized quantization and eight channels, total of 8 bpp. (c), (d) Best and worst images with four channels, total of 4 bpp. (e), (f) Best and worst images with two channels, total of 2 bpp. (g), (h) Best and worst images with one channel, total of 1 bpp.

## APPENDIX

**A. Proofs of Propositions.** We prove the various claims from the text. In the course of the proofs, we often find it convenient to express an integration over order statistics, i.e., with one set of variables, say $\{p_i\}$, equal to another unordered set of variables $\{q_i\}$, such that the $p_i$ must satisfy $p_1 < p_2 < \cdots < p_n$. When an integrand $f(\{q_i\})$ is a symmetric function of the $q_i$, then we will use the observation that

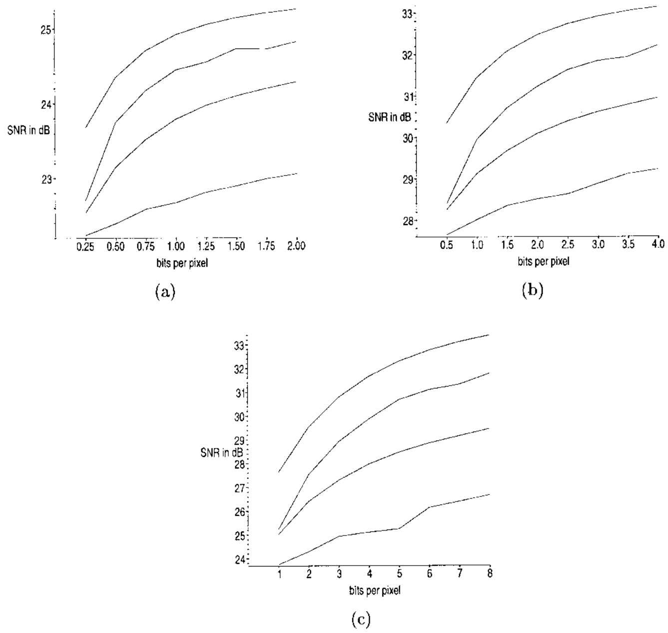


**Figure 13.** SNR values for the (a) Barbara image, (b) Goldhill image, and (c) scanned engraving as a function of the number of channels carrying the signal with randomized quantization. From top to bottom, the graphs are the expected SNR values according to Eq. (12) and the maximum, average, and minimum SNR values over 1,000 trials with one to eight channels receiving the signal. Each channel carries 1/4, 1/2, and 1 bpp in (a), (b), and (c), respectively.

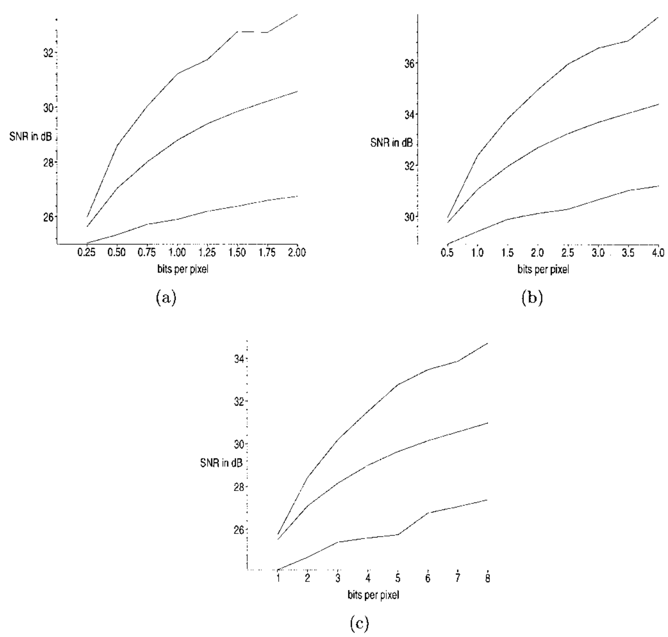


**Figure 14.** SNR values for the (a) Barbara image, (b) Goldhill image, and (c) scanned engraving compared with the images obtained by zeroing out the transformed pixels that are assigned no bits, with randomized quantization. From top to bottom, maximum, average, and minimum SNR values over 1,000 trials with one to eight channels receiving the signal. Each channel carries 0.25, 0.5, and 1 bpp in (a), (b), and (c), respectively.

$$\int_0^1 \int_0^1 \cdots \int_0^1 f(\{q_i\}) dq_1 \cdots dq_n$$

$$= n! \int_0^1 \int_0^{p_n} \cdots \int_0^{p_1} f(\{p_i\}) dp_1 \ldots dp_n.$$

This may be observed by breaking the original integral into $n!$ pieces ($n$-dimensional tetrahedra) according to the order of the $q_i$ and noting that the integrals over each region are all equal.

**Proposition 1:** *Let $S_j = \{j/n, (j + l)/n, (j + 2l)/n, \ldots, [j + (m - 2)l]/n\}$, $j = 1, \ldots, l$, where $n = (m - 1)l + 1$. Let $k$ distinct integers from $\{1, \ldots, l\}$ be chosen randomly and denote these by $a_j$, where $a_1 < a_2 < \cdots < a_k$. The points in $S_{a_1}, \ldots, S_{a_k}$ then divide the unit interval into $(m - 1)k + 1$ subintervals. A random variable $X$, distributed uniformly on the unit interval, is quantized to $(m - 1)k + 1$ values by mapping its realizations into the midpoints of the subintervals to which they belong. The expected (root mean square) error incurred by this quantization is given by*

$$\frac{\{[(k-1)(m-1)+2](l+1)[6l^2-6(k-2)l+(k-1)(k-6)] + (m-2)(2l-k+1)[12l^2-12(k-1)l+(k-1)(k-6)]\}^{1/2}}{2\sqrt{3}\,\sqrt{(k+1)(k+2)(k+3)}\,n^{3/2}}$$

**Proof:** The expected squared error in $X$ for a given set of $\{\mathbf{a}\} = \{a_1, \ldots, a_k\}$ is

$$E[\Delta X(\{\mathbf{a}\})]^2 = \int_0^{a_1/n} \left(x - \frac{a_1}{2n}\right)^2 dx$$

$$+ \sum_{i=0}^{m-2} \sum_{j=1}^{k-1} \int_{(a_j+il)/n}^{(a_{j+1}+il)/n} \left[x - \frac{a_j + a_{j+1} + 2il}{2n}\right]^2 dx$$

$$+ \sum_{i=0}^{m-3} \int_{(a_k+il)/n}^{[a_1+(i+1)l]/n} \left[x - \frac{a_k + a_1 + (2i+1)l}{2n}\right]^2 dx$$

$$+ \int_{[a_k+(m-2)l]/n}^{1} \left[x - \frac{a_k + n + (m-2)l}{2n}\right]^2 dx$$

$$= \frac{2}{3}\left[\left(\frac{a_1}{2n}\right)^3 + \sum_{i=0}^{m-2}\sum_{j=1}^{k-1}\left(\frac{a_{j+1} - a_j}{2n}\right)^3 + \sum_{i=0}^{m-3}\left(\frac{a_1 - a_k + l}{2n}\right)^3 + \left(\frac{n - a_k - (m-2)l}{2n}\right)^3\right]$$

$$= \frac{1}{12n^3}[a_1^3 + (m-1)\sum_{j=1}^{k-1}(a_{j+1} - a_j)^3 + (m-2)(a_1 - a_k + l)^3 + [n - a_k - (m-2)l]^3].$$

The expected squared error over all sets $\{\mathbf{a}\}$ is

$$E[\Delta X]^2 = \frac{\sum_{a_k=k}^{l} \sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} E[\Delta X(\{\mathbf{a}\})]^2}{\binom{l}{k}}. \qquad (13)$$

(Because the elements $a_j$ are in ascending order and $a_1 \geq 1$, we have $a_j \geq j$, hence the lower limits of summation in Eq. [13].)

We will require the use of several lemmas.

**Lemma 1:**

$$\sum_{a_k=k}^{l} \sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} [n - a_k - (m-2)l]^3 = \frac{(l+1)![6l^2 - 6(k-2)l + (k-1)(k-6)]}{(k+3)!(l-k)!}, \quad k \geq 1.$$

**Proof:** Because $n = (m-1)l + 1$, this is equivalent to

$$\sum_{a_k=k}^{l} \sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} (l+1-a_k)^3 = \frac{(l+1)!}{(k+3)!(l-k)!}[6l^2 - 6(k-2)l + (k-1)(k-6)]. \quad (14)$$

Because the first $k-1$ interior summations of Eq. (14) do not involve $a_k$, they can be evaluated directly. The inner summation

$$\sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} 1 = \binom{a_k - 1}{k-1}$$

because 1 is added to the sum for each subset of $\{a_1, \ldots a_{k-1}\}$ satisfying $1 \leq a_1 < a_2 < \cdots < a_{k-2} < a_{k-1} \leq a_k - 1$. This is the same as choosing $k-1$ elements from a set of cardinality $a_k - 1$. Consequently, we have

$$\sum_{a_k=k}^{l} \sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} (l+1-a_k)^3 = \sum_{a_k=k}^{l} \binom{a_k-1}{k-1}(l+1-a_k)^3$$

$$= \sum_{b=1}^{l-k+1} \binom{l-b}{k-1} b^3 \quad (15)$$

where we made the substitution $b = l + 1 - a_k$. We will prove Eq. (14) by introducing a series of sublemmas, each of which depends on those preceding it.

**Sublemma 1.1:**

$$\sum_{b=0}^{l-j} \binom{l-b}{j} = \binom{l+1}{j+1}, \quad 0 \leq j \leq l. \quad (16)$$

**Proof:** For $j = l$, this is

$$\sum_{b=0}^{0} \binom{l-b}{l} = 1.$$

Now assume Eq. (16) is true for $j = k$. Then for $j = k - 1$,

$$\sum_{b=0}^{l-k+1} \binom{l-b}{k-1} = \sum_{b=0}^{l-k+1} \left[ \binom{l-b+1}{k} - \binom{l-b}{k} \right]$$

$$= \left[ \sum_{b=1}^{l-k+1} \binom{l-b+1}{k} + \binom{l+1}{k} \right] - \sum_{b=0}^{l-k} \binom{l-b}{k}$$

$$= \sum_{b'=0}^{l-k} \binom{l-b'}{k} - \sum_{b=0}^{l-k} \binom{l-b}{k} + \binom{l+1}{k}$$

$$= \binom{l+1}{k}.$$

**Sublemma 1.2:**

$$\sum_{b=1}^{l-j} \binom{l-b}{j} b = \binom{l+1}{j+2}, \quad 0 \leq j \leq l-1. \quad (17)$$

**Proof:** For $j = l - 1$, this is

$$\sum_{b=1}^{1} \binom{l-b}{l-1} = 1.$$

Now, assume Eq. (17) is true for $j = k$. Then, for $j = k - 1$,

$$\sum_{b=1}^{l-k+1} \binom{l-b}{k-1} b = \sum_{b=1}^{l-k+1} \binom{l-b+1}{k} b - \sum_{b=1}^{l-k+1} \binom{l-b}{k} b$$

$$= \sum_{b'=0}^{l-k} \binom{l-b'}{k}(b'+1) - \sum_{b=1}^{l-k+1} \binom{l-b}{k} b$$

$$= \sum_{b'=0}^{l-k} \binom{l-b'}{k} b' + \sum_{b'=0}^{l-k} \binom{l-b'}{k} - \sum_{b=1}^{l-k} \binom{l-b}{k} b$$

$$= \binom{l+1}{k+1}.$$

**Sublemma 1.3:**

$$\sum_{b=1}^{l-j} \binom{l-b}{j} b^2 = \frac{(l+1)!(2l-j+1)}{(j+3)!(l-j-1)!}, \quad 0 \leq j \leq l-1. \quad (18)$$

**Proof:** For $j = l - 1$, this is

$$\sum_{b=1}^{1} \binom{l-b}{l-1} b^2 = \frac{(l+1)!(l+2)}{(l+2)!0!} = 1.$$

Now, assume Eq. (18) is true for $j = k$. Then, for $j = k - 1$,

$$\sum_{b=1}^{l-k+1}\binom{l-b}{k-1}b^2=\sum_{b=1}^{l-k+1}\binom{l-b+1}{k}b^2-\sum_{b=1}^{l-k+1}\binom{l-b}{k}b^2$$

$$=\sum_{b'=0}^{l-k}\binom{l-b'}{k}(b'+1)^2-\sum_{b=1}^{l-k+1}\binom{l-b}{k}b^2$$

$$=\sum_{b'=0}^{l-k}\binom{l-b'}{k}b'^2+2\sum_{b'=0}^{l-k}\binom{l-b'}{k}b'$$

$$+\sum_{b'=0}^{l-k}\binom{l-b'}{k}-\sum_{b=0}^{l-k}\binom{l-b}{k}b^2$$

$$=2\binom{l+1}{k+2}+\binom{l+1}{k+1}$$

$$=\frac{(l+1)!(2l-k+2)}{(k+2)!(l-k)!}.$$

**Sublemma 1.4:**

$$\sum_{b=1}^{l-j}\binom{l-b}{j}b^3=\frac{(l+1)![6l^2-6(j-1)l+j(j-5)]}{(j+4)!(l-j-1)!},$$

$$0\le j\le l-1. \quad (19)$$

**Proof:** For $j = l - 1$, this is

$$\sum_{b=1}^{1}\binom{l-b}{l-1}b^3=\frac{(l+1)!(l^2+5l+6)}{(l+3)!0!}=1.$$

Now, assume Eq. (19) is true for $j = k$. Then, for $j = k - 1$,

$$\sum_{b=1}^{l-k+1}\binom{l-b}{k-1}b^3=\sum_{b=1}^{l-k+1}\binom{l-b+1}{k}b^3-\sum_{b=1}^{l-k+1}\binom{l-b}{k}b^3$$

$$=\sum_{b'=0}^{l-k}\binom{l-b'}{k}(b'+1)^3-\sum_{b=1}^{l-k+1}\binom{l-b}{k}b^3$$

$$=\sum_{b'=0}^{l-k}\binom{l-b'}{k}b'^3+3\sum_{b'=0}^{l-k}\binom{l-b'}{k}b'^2$$

$$+3\sum_{b'=0}^{l-k}\binom{l-b'}{k}b'+\sum_{b'=0}^{l-k}\binom{l-b'}{k}-\sum_{b=0}^{l-k}\binom{l-b}{k}b^3$$

$$=3\frac{(l+1)!(2l-k+1)}{(k+3)!(l-k-1)!}+3\binom{l+1}{k+2}+\binom{l+1}{k+1}$$

$$=\frac{(l+1)![6l^2-6(k-2)l+(k-1)(k-6)]}{(k+3)!(l-k)!}.$$

**Completion of Proof of Lemma 1:** Replacing $k$ by $k - 1$ in Eq. (19) and plugging into Eq. (15) produces the statement of Lemma 1.

**Lemma 2:**

$$\sum_{a_k=k}^{l}\sum_{a_{k-1}=k-1}^{a_k-1}\cdots\sum_{a_1=1}^{a_2-1}a_1^3$$
$$=\frac{(l+1)![6l^2-6(k-2)l+(k-1)(k-6)]}{(k+3)!(l-k)!},\quad k\ge 1. \quad (20)$$

We have

$$\sum_{a_k=k}^{l}\sum_{a_{k-1}=k-1}^{a_k-1}\cdots\sum_{a_1=1}^{a_2-1}a_1^3=\sum_{b_k=1}^{l-k+1}\sum_{b_{k-1}=b_k+1}^{l-k+2}\cdots\sum_{b_1=b_2+1}^{l}(l+1-b_1)^3$$

$$=\sum_{a_1=1}^{l-k+1}\sum_{a_2=a_1+1}^{l-k+2}\cdots\sum_{a_k=a_{k-1}+1}^{l}(l+1-a_k)^3$$

$$=\sum_{a_k=k}^{l}\sum_{a_{k-1}=k-1}^{a_k-1}\cdots\sum_{a_1=1}^{a_2-1}(l+1-a_k)^3$$

$$=\frac{(l+1)![6l^2-6(k-2)l+(k-1)(k-6)]}{(k+3)!(l-k)!},$$

where the first equality is obtained by the substitutions $a_i \to l + 1 - b_i$, the second equality by the substitutions $b_i \to a_{k-i+1}$, the third equality by changing the order of summation, and the fourth equality from Lemma 1.

**Lemma 3:**

$$\sum_{a_k=k}^{l}\sum_{a_{k-1}=k-1}^{a_k-1}\cdots\sum_{a_1=1}^{a_2-1}(a_{j+1}-a_j)^3$$
$$=\frac{(l+1)![6l^2-6(k-2)l+(k-1)(k-6)]}{(k+3)!(l-k)!},$$
$$1\le j\le k-1. \quad (21)$$

**Proof:** When $j = 1$, the innermost summation of Eq. (21) is

$$\sum_{a_1=1}^{a_2-1}(a_2-a_1)^3=\frac{(a_2-1)^2a_2^2}{4}.$$

This is the sum of the first $a_2 - 1$ positive cubes and is the same as the innermost summation of Eq. (20). Therefore, the entire summation has the same value as the multiple summation in Eq. (20), which is

$$\frac{(l+1)![6l^2-6(k-2)l+(k-1)(k-6)]}{(k+3)!(l-k)!}.$$

We now show that

$$\sum_{a_k=k}^{l}\sum_{a_{k-1}=k-1}^{a_k-1}\cdots\sum_{a_1=1}^{a_2-1}(a_{j+2}-a_{j+1})^3$$

$$=\sum_{a_k=k}^{l}\sum_{a_{k-1}=k-1}^{a_k-1}\cdots\sum_{a_1=1}^{a_2-1}(a_{j+1}-a_j)^3, \quad (22)$$

for $1 \le j \le k-2$. Because the first $j-1$ interior summations of each expression in Eq. (22) do not involve $a_j$, $a_{j+1}$, or $a_{j+2}$, they can be evaluated directly, and thus showing Eq. (22) is equivalent to showing

$$\sum_{a_k=k}^{l}\sum_{a_{k-1}=k-1}^{a_k-1}\cdots\sum_{a_j=j}^{a_{j+1}-1}\binom{a_j-1}{j-1}(a_{j+2}-a_{j+1})^3$$

$$=\sum_{a_k=k}^{l}\sum_{a_{k-1}=k-1}^{a_k-1}\cdots\sum_{a_j=j}^{a_{j+1}-1}\binom{a_j-1}{j-1}(a_{j+1}-a_j)^3, \quad (23)$$

(The inner summation

$$\sum_{a_{j-1}=j-1}^{a_j-1}\cdots\sum_{a_1=1}^{a_2-1}1=\binom{a_j-1}{j-1}$$

because 1 is added to the sum for each subset of $\{a_1, \ldots a_{j-1}\}$ satisfying $1 \le a_1 < a_2 < \cdots < a_{j-2} < a_{j-1} \le a_j - 1$. This is the same as choosing $j-1$ elements from a set of cardinality $a_j - 1$.)

Equation (23) will be proved by showing that the results of the innermost two summations in each side of Eq. (23) are the same function of $a_{j+2}$. To wit,

$$\sum_{a_{j+1}=j+1}^{a_{j+2}-1}\sum_{a_j=j}^{a_{j+1}-1}\binom{a_j-1}{j-1}(a_{j+2}-a_{j+1})^3$$

$$=\sum_{a_{j+1}=j+1}^{a_{j+2}-1}\binom{a_{j+1}-1}{j}(a_{j+2}-a_{j+1})^3,$$

and

$$\sum_{a_{j+1}=j+1}^{a_{j+2}-1}\sum_{a_j=j}^{a_{j+1}-1}\binom{a_j-1}{j-1}(a_{j+1}-a_j)^3$$

$$=\sum_{a_j=j}^{a_{j+2}-1}\sum_{a_{j+1}=a_j+1}^{a_{j+2}-1}\binom{a_j-1}{j-1}(a_{j+1}-a_j)^3$$

$$=\sum_{a_j=j}^{a_{j+2}-1}\binom{a_j-1}{j-1}\frac{(a_{j+2}-a_j-1)^2(a_{j+2}-a_j)^2}{4}$$

$$=\sum_{a_j=j}^{a_{j+2}-1}\left[\binom{a_j}{j}-\binom{a_j-1}{j}\right]$$

$$\times\left[\frac{(a_{j+2}-a_j)^2(a_{j+2}-a_j+1)^2}{4}-(a_{j+2}-a_j)^3\right]$$

$$=\sum_{a_j=j}^{a_{j+2}-1}\binom{a_j}{j}\frac{(a_{j+2}-a_j)^2(a_{j+2}-a_j+1)^2}{4}$$

$$-\sum_{a_j=j}^{a_{j+2}-1}\binom{a_j}{j}(a_{j+2}-a_j)^3-\sum_{a_j=j+1}^{a_{j+2}}\binom{a_j-1}{j}$$

$$\times\frac{(a_{j+2}-a_j)^2(a_{j+2}-a_j+1)^2}{4}+\sum_{a_j=j}^{a_{j+2}-1}\binom{a_j-1}{j}(a_{j+2}-a_j)^3$$

$$=\sum_{a_j=j}^{a_{j+2}-1}\binom{a_j}{j}\frac{(a_{j+2}-a_j)^2(a_{j+2}-a_j+1)^2}{4}-\sum_{a_j=j}^{a_{j+2}-1}\binom{a_j}{j}$$

$$\times(a_{j+2}-a_j)^3-\sum_{a_j'=j}^{a_{j+2}-1}\binom{a_j'}{j}\frac{(a_{j+2}-a_j'-1)^2(a_{j+2}-a_j')^2}{4}$$

$$+\sum_{a_j=j}^{a_{j+2}-1}\binom{a_j-1}{j}(a_{j+2}-a_j)^3$$

$$=\sum_{a_j=j}^{a_{j+2}-1}\binom{a_j}{j}\left[\frac{(a_{j+2}-a_j)^2(a_{j+2}-a_j+1)^2}{4}-(a_{j+2}-a_j)^3\right.$$

$$\left.-\frac{(a_{j+2}-a_j-1)^2(a_{j+2}-a_j)^2}{4}\right]$$

$$+\sum_{a_j=j}^{a_{j+2}-1}\binom{a_j-1}{j}(a_{j+2}-a_j)^3$$

$$=\sum_{a_j=j}^{a_{j+2}-1}\binom{a_j-1}{j}(a_{j+2}-a_j)^3.$$

The last expressions in the above two chains of equalities are equal, having only a different index of summation. Consequently, the two double sums are equal as are the values of the entire multiple summations in Eq. (23).

**Lemma 4:**

$$\sum_{a_k=k}^{l}\sum_{a_{k-1}=k-1}^{a_k-1}\cdots\sum_{a_1=1}^{a_2-1}(a_1-a_k+l)^3$$

$$=\frac{l!(2l-k+1)[12l^2-12(k-1)l+(k-1)(k-6)]}{(k+3)!(l-k)!},$$

$$k \ge 1. \quad (24)$$

**Proof:** We write $(a_1 - a_k + l)^3$ as $[a_1 + (l - a_k)]^3$. Then, we have

$$\sum_{a_k=k}^{l}\sum_{a_{k-1}=k-1}^{a_k-1}\cdots\sum_{a_1=1}^{a_2-1}a_1^3$$

$$=\frac{(l+1)![6l^2-6(k-2)l+(k-1)(k-6)]}{(k+3)!(l-k)!} \quad (25)$$

from Lemma 2 and

$$\sum_{a_k=k}^{l} \sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} (l-a_k)^3 = \frac{l![6l^2 - 6kl + (k-1)k]}{(k+3)!(l-k-1)!} \tag{26}$$

by replacing $l$ with $l-1$ in Lemma 1.

**Sublemma 4.1:**

$$\sum_{a_k=k}^{l} \sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} a_1(l-a_k)^2 = \frac{(l+1)!(2l-k+1)}{(k+3)!(l-k-1)!}, \quad k \geq 1. \tag{27}$$

Because the first $k-1$ interior summations of Eq. (27) do not involve $a_k$, we only have to consider the contribution from $a_1$. The result is

$$\sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} a_1 = \binom{a_k}{k}.$$

This is a standard binomial coefficient identity, with the induction step

$$\sum_{a_k=k}^{a_{k+1}-1} \binom{a_k}{k} = \binom{a_{k+1}}{k+1}.$$

This is recognized as a way of breaking up the selection of $k+1$ integers from $[1, a_k+1]$ as choosing the largest (one of $k+1, \ldots, a_{k+1}$) and then choosing the other $k$ from the corresponding interval ($[1, k], \ldots, [1, a_{k+1}-1]$). Thus, we now need to show that

$$\sum_{a_k=k}^{l} \binom{a_k}{k}(l-a_k)^2 = \frac{(l+1)!(2l-k+1)}{(k+3)!(l-k-1)!}.$$

However, this is Sublemma 1.3 with the index $b$ replaced by $l-a_k$.

**Sublemma 4.2:**

$$\sum_{a_k=k}^{l} \sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} a_1^2(l-a_k) = \frac{(l+1)!(2l-k+1)}{(k+3)!(l-k-1)!}, \quad k \geq 1. \tag{28}$$

**Proof:** Using a technique similar to that of the proof of Lemma 2, we have

$$\sum_{a_k=k}^{l} \sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} a_1^2(l-a_k) = \sum_{b_k=0}^{l-k} \sum_{b_{k-1}=b_k+1}^{l-k+1} \cdots \sum_{b_1=b_2+1}^{l-1} (l-b_1)^2 b_k$$

$$= \sum_{a_1=0}^{l-k} \sum_{a_2=a_1+1}^{l-k+1} \cdots \sum_{a_k=a_{k-1}+1}^{l-1} (l-a_k)^2 a_1$$

$$= \sum_{a_k=k}^{l-1} \sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=0}^{a_2-1} a_1(l-a_k)^2$$

$$= \sum_{a_k=k}^{l} \sum_{a_{k-1}=k-1}^{a_k-1} \cdots \sum_{a_1=1}^{a_2-1} a_1(l-a_k)^2$$

$$= \frac{(l+1)!(2l-k+1)}{(k+3)!(l-k-1)!},$$

where the first equality is obtained by the substitutions $a_i \to l+1-b_i$, the second by the substitutions $b_i \to a_{k-i+1}$, the third by changing the order of summation, the fourth by recognizing that terms with $a_1 = 0$ or $a_k = l$ contribute zero to the sum of $a_1(l-a_k)^2$, and the fifth from Sublemma 4.1.

**Completion of Proof of Lemma 4:** Combining Eqs. (25), (26), (27), and (28), including factors of 3 for Eqs. (27) and (28), yields the statement of Eq. (24).

**Completion of Proof of Proposition 1:** There is one term each from Lemmas 1 and 2, $(m-1)(k-1)$ terms from Lemma 3, and $m-2$ terms from Lemma 4. Combining these with Eq. (13) yields that the expected squared error over all sequences $\{\mathbf{a}\}$ is

$$E[\Delta X]^2 = \frac{\dfrac{[(k-1)(m-1)+2](l+1)! \times [6l^2 - 6(k-2)l + (k-1)(k-6)]}{(k+3)!(l-k)!} + \dfrac{(m-2)l!(2l-k+1) \times [12l^2 - 12(k-1)l + (k-1)(k-6)]}{(k+3)!(l-k)!}}{\dbinom{l}{k}}$$

$$= \frac{[(k-1)(m-1)+2](l+1) \times [6l^2 - 6(k-2)l + (k-1)(k-6)] + (m-2)(2l-k+1) \times [12l^2 - 12(k-1)l + (k-1)(k-6)]}{12(k+1)(k+2)(k+3)n^3}.$$

**Proposition 2:** *Suppose $n-1$ random points are i. i. d. uniformly on the unit interval. Let the ordered set of these points be denoted by $\{\mathbf{p}\} = \{p_1, \ldots, p_{n-1}\}$ where $0 \leq p_1 \leq p_2 \leq \ldots \leq p_{n-1} \leq 1$. These points define $n$ subintervals $[0, p_1)$, $[p_1, p_2)$, $\ldots$, $[p_{n-2}, p_{n-1})$, $[p_{n-1}, 1]$. A random variable $X$, also uniformly distributed on the unit interval, will be quantized to $n$ values by mapping its realizations $x$ into $Q(x)$, the midpoints of the subintervals to which they belong. The expected (root mean square) error incurred by this quantization is given by*

$$\frac{1}{\sqrt{2(n+1)(n+2)}}.$$

**Proof:** The expected squared error in $X$ for a given set of points $\{\mathbf{p}_i\}$ is

$$E[\Delta X(\{\mathbf{p}\})]^2 = \int_0^1 [x - Q(x)]^2\, dx$$

$$= \int_0^{p_1} \left(x - \frac{p_1}{2}\right)^2 dx + \sum_{i=1}^{n-2} \int_{p_i}^{p_{i+1}} \left(x - \frac{p_i + p_{i+1}}{2}\right)^2 dx$$

$$+ \int_{p_{n-1}}^1 \left(x - \frac{1 + p_{n-1}}{2}\right)^2 dx$$

$$= \frac{1}{12}\left[p_1^3 + \sum_{i=1}^{n-2} (p_{i+1} - p_i)^3 + (1 - p_{n-1})^3\right].$$

We now need several more lemmas.

**Lemma 5:**

$$\int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_2} (1 - p_{n-1})^3 dp_1 \ldots dp_{n-2}\, dp_{n-1} = \frac{6}{(n+2)!}, \quad n \geq 2. \quad (29)$$

**Proof:** Because the first $n - 2$ interior integrals of Eq. (29) do not involve $p_{n-1}$, they can be evaluated directly. Because

$$\int_0^{p_{n-1}} \cdots \int_0^{p_2} 1 dp_1 \ldots dp_{n-2} = \frac{p_{n-1}^{n-2}}{(n-2)!},$$

we have

$$\int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_2} (1 - p_{n-1})^3 dp_1 \ldots dp_{n-2} dp_{n-1}$$

$$= \int_0^1 \frac{p_{n-1}^{n-2}}{(n-2)!} (1 - p_{n-1})^3 dp_1 = \frac{6}{(n+2)!}.$$

**Lemma 6:**

$$\int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_2} p_1^3 dp_1 \ \ldots dp_{n-2}\, dp_{n-1} = \frac{6}{(n+2)!}, \quad n \geq 2. \quad (30)$$

**Proof:** Using a technique analogous to that for the proof of Lemma 2 but for integrals instead of sums, we have

$$\int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_2} p_1^3 dp_1 \ldots dp_{n-2}\, dp_{n-1}$$

$$= \int_0^1 \int_{q_{n-1}}^1 \cdots \int_{q_2}^1 (1 - q_1)^3 dq_1 \ldots dq_{n-2}\, dq_{n-1}$$

$$= \int_0^1 \int_{p_1}^1 \cdots \int_{p_{n-2}}^1 (1 - p_{n-1})^3 dp_{n-1} \ldots dp_2\, dp_1$$

$$= \int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_2} (1 - p_{n-1})^3 dp_1 \ldots dp_{n-2}\, dp_{n-1}$$

$$= \frac{6}{(n+2)!},$$

where the first equality is obtained by the substitutions $p_i \to 1 - q_i$, the second equality by the substitutions $q_i \to p_{n-i}$, the third equality by changing the order of integration and recognizing the consequent changes to the limits of integration, and the fourth equality from Lemma 5.

**Lemma 7:**

$$\int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_2} (p_{i+1} - p_i)^3 dp_1 \ldots dp_{n-2}\, dp_{n-1} = \frac{6}{(n+2)!}, \quad n \geq 3, \quad 1 \leq i \leq n-2. \quad (31)$$

**Proof:** When $i = 1$, the innermost integral of Eq. (31) is

$$\int_0^{p_2} (p_2 - p_1)^3 dp_1 = \frac{p_2^4}{4},$$

which is the same values as the innermost integral of Eq. (30). Therefore, the entire multiple integral has the same value as the multiple integral in Eq. (30), $6/(n + 2)!$.

We now show that

$$\int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_2} (p_{i+2} - p_{i+1})^3 dp_1 \ldots dp_{n-2}\, dp_{n-1}$$

$$= \int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_2} (p_{i+1} - p_i)^3 dp_1 \ldots dp_{n-2}\, dp_{n-1}, \quad (32)$$

for $1 \leq i \leq n - 2$. Because the first $i - 1$ interior integrals of each expression in Eq. (32) do not involve $p_i$, $p_{i+1}$, and $p_{i+2}$, they can be evaluated directly, and thus showing Eq. (32) is equivalent to showing

$$\int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_{i+1}} \frac{p_i^{i+1}}{(i-1)!} (p_{i+2} - p_{i+1})^3 dp_i \ldots dp_{n-2}\, dp_{n-1}$$

$$= \int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_{i+1}} \frac{p_i^{i+1}}{(i-1)!} (p_{i+1} - p_i)^3 dp_i \ldots dp_{n-2}\, dp_{n-1}. \tag{33}$$

This equation will be proved by showing that the results of the innermost two integrations in each side of Eq. (33) are the same function of $p_{i+2}$. To wit,

$$\int_0^{p_{i+2}} \int_0^{p_{i+1}} \frac{p_i^{i-1}}{(i-1)!} (p_{i+2} - p_{i+1})^3 dp_i\, dp_{i+1}$$

$$= \int_0^{p_{i+2}} \frac{p_{i+1}^i}{i!} (p_{i+2} - p_{i+1})^3 dp_{i+1},$$

and

$$\int_0^{p_{i+2}} \int_0^{p_{i+1}} \frac{p_i^{i-1}}{(i-1)!} (p_{i+2} - p_{i+1})^3 dp_i\, dp_{i+1}$$

$$= \int_0^{p_{i+2}} \int_{p_i}^{p_{i+2}} \frac{p_i^{i-1}}{(i-1)!} (p_{i+2} - p_{i+1})^3 dp_{i+1}\, dp_i$$

$$= \int_0^{p_{i+2}} \frac{p_i^{i-1}}{(i-1)!} \frac{(p_{i+2} - p_i)^4}{4}\, dp_i$$

$$= \left[\frac{p_i^i}{i!} \frac{(p_{i+2} - p_i)^4}{4}\right]_{p_i=0}^{p_i=p_{i+2}} + \int_0^{p_{i+2}} \frac{p_i^i}{i!} (p_{i+2} - p_i)^3\, dp_i.$$

The last step was obtained with an integration by parts and the term in brackets in the last line vanishes. Thus, the two double integrals are equal as are the values of the entire multiple integrals in Eq. (33).

**Completion of proof of Proposition 2:** There is one term each from Lemmas 5 and 6 and $n - 2$ terms from Lemma 7. Thus, the expected squared error over all sets of ordered points $\{\mathbf{p}_i\}$ is

$$E[\Delta X]^2 = \frac{\int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_2} E[\Delta X(\{\mathbf{p}\})]^2 dp_1 \ldots dp_{n-2} dp_{n-1}}{\int_0^1 \int_0^{p_{n-1}} \cdots \int_0^{p_2} 1 dp_1 \ldots dp_{n-2} dp_{n-1}}$$

$$= \frac{1}{12} \frac{\left[\frac{6}{(n+2)!} + \frac{6}{(n+2)!} + (n-2)\frac{6}{(n+2)!}\right]}{\frac{1}{(n-1)!}}$$

$$= \frac{\frac{6n}{(n+2)!}}{\frac{12}{(n-1)!}} = \frac{1}{2(n+1)(n+2)}.$$

**Proposition 3:** *For a given integer $m > 1$, define $s_i = (2i - 1)/[2(m-1)]$, $i = 1, \ldots, m-1$. Let $\epsilon_1, \ldots, \epsilon_k$ be chosen randomly and uniformly from the interval $(-1/[2(m-1)], 1/[2(m-1)])$ and order them so that $\epsilon_1 \le \epsilon_2 \le \cdots \le \epsilon_k$. Then, the points $s_i + \epsilon_j$ divide the unit interval into $k(m-1) + 1$* subintervals. A random variable *X*, *uniformly distributed on the unit interval, is quantized to $k(m-1) + 1$ values by mapping its realizations into the midpoints of the subintervals to which they belong. The expected (root mean square) error incurred by this quantization is given by*

$$\frac{\sqrt{(k+3)(m-1) - 2}}{\sqrt{2(k+1)(k+2)(k+3)}(m-1)^{3/2}}.$$

**Proof:** The expected squared error in $X$ for a given set of $\{\epsilon\} = \{\epsilon_1, \ldots, \epsilon_k\}$ is

$$E[\Delta X(\{\epsilon\})]^2 = \int_0^{1/(2m-2)+\epsilon_1} \left[x - \left(\frac{1}{4m-4} + \frac{\epsilon_1}{2}\right)\right]^2 dx$$

$$+ \sum_{i=1}^{m-1} \sum_{j=1}^{k-1} \int_{(2i-1)/(2m-2)+\epsilon_j}^{(2i-1)/(2m-2)+\epsilon_{j+1}} \left[x - \left(\frac{2i-1}{2m-2} + \frac{\epsilon_j + \epsilon_{j+1}}{2}\right)\right]^2 dx$$

$$+ \sum_{i=1}^{m-2} \int_{(2i-1)/(2m-2)+\epsilon_k}^{(2i+1)/(2m-2)+\epsilon_1} \left[x - \left(\frac{i}{m-1} + \frac{\epsilon_k + \epsilon_1}{2}\right)\right]^2 dx$$

$$+ \int_{(2m-3)/(2m-2)+\epsilon_k}^{1} \left[x - \left(\frac{4m-5}{4m-4} + \frac{\epsilon_k}{2}\right)\right]^2 dx$$

$$= \frac{2}{3}\left[\left(\frac{1}{4m-4} + \frac{\epsilon_1}{2}\right)^3 + \sum_{i=1}^{m-1} \sum_{j=1}^{k-1} \left(\frac{\epsilon_{j+1} - \epsilon_j}{2}\right)^3\right.$$

$$\left. + \sum_{i=1}^{m-2} \left(\frac{1}{2m-2} + \frac{\epsilon_1 - \epsilon_k}{2}\right)^3 + \left(\frac{1}{4m-4} - \frac{\epsilon_k}{2}\right)^3\right]$$

$$= \frac{1}{12}\left[\left(\frac{1}{2m-2} + \epsilon_1\right)^3 + (m-1)\sum_{j=1}^{k-1} (\epsilon_{j+1} - \epsilon_j)^3\right.$$

$$\left. + (m-2)\left(\frac{1}{m-1} + \epsilon_1 - \epsilon_k\right)^3 + \left(\frac{1}{2m-2} - \epsilon_k\right)^3\right].$$

The expected squared error over all sets $\{\epsilon\}$ is

$$E[\Delta X]^2 = \frac{\int_{-1/(2m-2)}^{1/(2m-2)} \int_{-1/(2m-2)}^{\epsilon_k} \cdots \int_{-1/(2m-2)}^{\epsilon_2} E[\Delta X(\{\epsilon\})]^2 d\epsilon_1 \ldots d\epsilon_{k-1} d\epsilon_k}{\int_{-1/(2m-2)}^{1/(2m-2)} \int_{-1/(2m-2)}^{\epsilon_k} \cdots \int_{-1/(2m-2)}^{\epsilon_2} 1 d\epsilon_1 \ldots d\epsilon_{k-1} d\epsilon_k}$$

$$= \frac{\int_0^{1/(m-1)} \int_0^{\zeta_k} \cdots \int_0^{\zeta_2} \left[ \begin{array}{l} \zeta_1^3 + (m-1) \sum_{j=1}^{k-1} (\zeta_{j+1} - \zeta_j)^3 \\ \quad + (m-2)\left(\frac{1}{m-1} + \zeta_1 - \zeta_k\right)^3 \\ \quad + \left(\frac{1}{m-1} - \zeta_k\right)^3 \end{array} \right] d\zeta_1 \ldots d\zeta_{k-1} d\zeta_k}{12 \int_0^{1/(m-1)} \int_0^{\zeta_k} \cdots \int_0^{\zeta_2} 1 \, d\zeta_1 \ldots d\zeta_{k-1} \, d\zeta_k}$$

$$= \frac{\frac{1}{(m-1)^{k+3}} \int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} \left[ \begin{array}{l} \eta_1^3 + (m-1) \sum_{j=1}^{k-1} (\eta_{j+1} - \eta_j)^3 \\ \quad + (m-2)(1 + \eta_1 - \eta_k)^3 \\ \quad + (1 - \eta_k)^3 \end{array} \right] d\eta_1 \ldots d\eta_{k-1} \, d\eta_k}{\frac{12}{(m-1)^k} \int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} 1 \, d\eta_1 \ldots d\eta_{k-1} \, d\eta_k}$$

$$= \frac{k!}{12(m-1)^3} \int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} \left[ \begin{array}{l} \eta_1^3 + (m-1) \sum_{j=1}^{k-1} (\eta_{j+1} - \eta_j)^3 \\ \quad + (m-2)(1 + \eta_1 - \eta_k)^3 \\ \quad + (1 - \eta_k)^3 \end{array} \right] d\eta_1 \ldots d\eta_{k-1} \, d\eta_k. \quad (34)$$

Three of the four terms inside the brackets in the last line of Eq. (34) can be handled by Lemmas 1, 2, and 3. Thus, we have

$$\int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} \eta_1^3 \, d\eta_1 \ldots d\eta_{k-1} \, d\eta_k = \frac{6}{(k+3)!},$$

$$\int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} (1 - \eta_k)^3 \, d\eta_1 \ldots d\eta_{k-1} \, d\eta_k = \frac{6}{(k+3)!},$$

and

$$\int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} (m-1) \sum_{j=1}^{k-1} (\eta_{j+1} - \eta_j)^3 \, d\eta_1 \ldots d\eta_{k-1} \, d\eta_k = \frac{6(m-1)(k-1)}{(k+3)!}.$$

We now need

**Lemma 8:**

$$\int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} (1 + \eta_1 - \eta_k)^3 \, d\eta_1 \ldots d\eta_{k-1} \, d\eta_k = \frac{24}{(k+3)!}. \quad (35)$$

**Proof:** We write $(1 + \eta_1 - \eta_k)^3$ as $[\eta_1 + (1 - \eta_k)]^3$. Then, we have

$$\int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} \eta_1^3 \, d\eta_1 \ldots d\eta_{k-1} \, d\eta_k = \frac{6}{(k+3)!}$$

from Lemma 2 and

$$\int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} (1 - \eta_k)^3 \, d\eta_1 \ldots d\eta_{k-1} \, d\eta_k = \frac{6}{(k+3)!}$$

from Lemma 1. Next,

$$\int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} \eta_1^2(1 - \eta_k) \, d\eta_1 \ldots d\eta_{k-1} \, d\eta_k = \int_0^1 \frac{\eta_k^{k+1}(1 - \eta_k)}{(k+1)!} \, d\eta_k = \frac{2}{(k+3)!},$$

and by the previous set of substitutions,

$$\int_0^1 \int_0^{\eta_k} \cdots \int_0^{\eta_2} \eta_1(1 - \eta_k)^2 \, d\eta_1 \cdots d\eta_{k-1} \, d\eta_k = \frac{2}{(k+3)!}.$$

Combining these last four equations (including factors of 3 for the $\eta_1 \eta_k^2$ and $\eta_1^2 \eta_k$ terms) yields Eq. (35).

**Completion of proof of Proposition 3:** Multiplying Eq. (35) by $m - 2$ and adding to the other terms in the brackets in Eq. (34) give the result

$$E[\Delta X]^2$$

$$= \frac{k!}{12}\left[\frac{6}{(k+3)!} + \frac{6}{(k+3)!} + \frac{6(m-1)(k-1)}{(k+3)!} + \frac{24(m-2)}{(k+3)!}\right]$$

$$= \frac{(k+3)(m-1)-2}{2(k+1)(k+2)(k+3)(m-1)^3}.$$

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