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Fast Matrix Factorizations via Discrete Transmission Lines*

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ABSTRACT

Fast algorithms, associated with the names of Schur and Levinson, are known for the triangular factorization of symmetric, positive definite Toeplitz matrices and their inverses. In this paper we show that these algorithms can be derived from simple arguments involving causality, symmetry, and energy conservation in discrete lossless transmission lines. The results not only provide a nice interpretation of the classical Schur and Levinson algorithms and a certain Toeplitz inversion formula of Gohberg and Semencul, but they also show immediately that the same fast algorithms apply not only to Toeplitz matrices but to all matrices with so-called displacement inertia $(1,1)$. The results have been helpful in suggesting new digital filter structures and in the study of nonstationary second-order processes.

1. INTRODUCTION

The aim of this paper is to show that a simple physical device, known to electrical engineers as a discrete transmission line (i.e. one with piecewise constant impedance profile) and to geophysicists as a "layered earth" model (see e.g. [3], [6], [15]), can be used to easily derive fast algorithms for the triangular factorization of matrices with a certain structure. The need for the factorization of a symmetric, positive definite matrix $\mathbf{R}$ into the product of

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(lower and upper) triangular factors arises in many contexts. This so-called Cholesky factorization is unique if the diagonal elements of the factors are arranged to be positive, as can always be done for positive definite matrices. There are several algorithms for Cholesky factorization (see e.g. [9]), and all of them require $O(N^3)$ elementary operations (additions or multiplications of two real numbers) for the factorization of a general $N \times N$ matrix.

In many applications the dimension N can be quite large, and therefore there is interest in determining special classes of matrices for which the computational effort can be significantly reduced. It was shown by Bareiss [2], Morf [12], Rissanen [14], and others that symmetric, positive definite Toeplitz matrices, i.e. matrices of the form

$$\mathbf{T}_N = [c_{i-j}]_{i,j=0}^{N} \tag{1.1}$$

can be factored with $O(N^2)$ elementary operations. This is an important class of matrices; for example, covariance matrices of stationary Gaussian random sequences are Toeplitz. The so-called fast Cholesky algorithms obtained in the above references are slightly different, but the derivations are all algebraic and involve considerable manipulation.

In this paper we shall show that a discrete, nonuniform, and lossless transmission-line model can be used to obtain a simple graphical description and derivation of the fast Cholesky algorithm for Toeplitz matrices. Moreover the transmission-line derivation almost immediately yields the interesting and useful result that the same fast algorithm can be used to factor *any* positive definite matrix of the form

$$\mathbf{R}_N = L(U_N)L^T(U_N) - L(V_N)L^T(V_N), \tag{1.2}$$

where $L(U)$ denotes a lower triangular Toeplitz matrix with first column equal to the vector U. Symmetric Toeplitz matrices are a special case of (1.2) arising from the identity

$$\mathbf{T}_N = L(E_N + C_N)L^T(E_N + C_N) - L(C_N)L^T(C_N), \tag{1.3}$$

where $E_N = [1\ 0\ 0\ \cdots\ 0]^T$ and $C_N = [0\ c_1\ c_2\ \cdots\ c_N]^T$ (and $c_0 = 1$).

We shall also show that the general fast algorithm is in fact completely equivalent to a recursion given by I. Schur in [16] for checking if a power series in z is bounded inside the unit disc $|z| \leqslant 1$ (see also [1, p. 101]). The paper of Schur also contains an identity that sheds more light on the extension from Toeplitz matrices to matrices of the form (1.2). This identity shows that any such matrix $\mathbf{R}_N$ is in fact congruent to a symmetric Toeplitz matrix, and the congruence matrix Λ_N is lower triangular Toeplitz and given by

$$\Lambda_N = L(U_N - V_N). \tag{1.4}$$

These results motivate us to explore the transmission-line structure further, and this leads to the discovery that the line can also be used to obtain a fast algorithm for factoring inverses of matrices of the type (1.2). Moreover these arguments also lead to the result that

$$\mathbf{R}_N^{-1} = L(F_N)L^T(F_N) - L(G_N)L^T(G_N), \tag{1.5}$$

which may be recognized as a special case of a closure theorem for matrices with a so-called displacement structure.

An Outline of Results

The basic results of this paper are simple enough to be described here. If $U_N = [u_0, u_1, \ldots, u_N]^T$ and $V_N = [v_0, v_1, \ldots, v_N]^T$ are given sequences for which (1.2) is a positive definite matrix (and $v_0 = 0$), we shall show that we can always set up a discrete transmission line as shown in Figure 1, so that the sequence defined by V_N will be the causal response of the line (starting at rest) to the input sequence U_N. Let

$x_i(j)$ = the value appearing at time j at the input of the ith delay element in Figure 1.

Note that, by causality of the signal flow, we have

$$x_i(j) = 0 \qquad \text{for} \quad j < i. \tag{1.6}$$

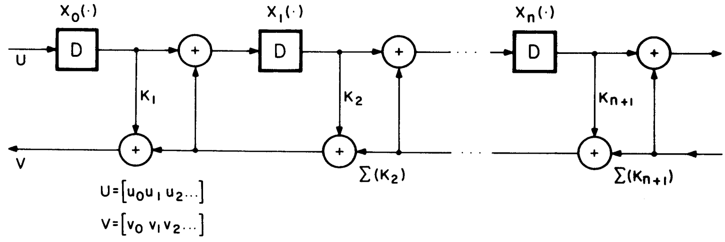

FIG. 1. Discrete transmission line (scattering representation).

Then the matrix

$$\mathbf{X}=\begin{bmatrix} x_0(0) & 0 & 0 & \cdots & 0 \\ x_0(1) & x_1(1) & 0 & \cdots & 0 \\ x_0(2) & x_1(2) & x_2(2) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x_0(N) & x_1(N) & x_2(N) & \cdots & x_N(N) \end{bmatrix} \tag{1.7}$$

will be the lower triangular Cholesky factor of the matrix given by (1.2), i.e.,

$$\mathbf{R}_N = L(U_N)L^T(U_N) - L(V_N)L^T(V_N)L = \mathbf{X}\mathbf{X}^T. \tag{1.8}$$

In other words, the history of the input to the ith delay element is the ith column of the Cholesky factor of $\mathbf{R}_N$. A simple energy-conservation justification of the identity (1.8) will be given in Section 3.

The transmission line structure of Figure 1 can be modified so as to reverse the direction of the signal flow on the lower line, while preserving the relationships between the values of the inputs and outputs of each section. The resulting lattice structure is shown in Figure 2, and a discussion of its properties is given in Section 4. If the resulting forward-propagation system is excited with the sequence [1 0 0 0 $\cdots$ 0] at both input points, then denoting

$y_i(j)$ = the value appearing at time j at the input of the ith delay element in Figure 2,

we have that the signal flow dictates

$$y_i(j) = 0 \qquad \text{for} \quad j > i \tag{1.9}$$

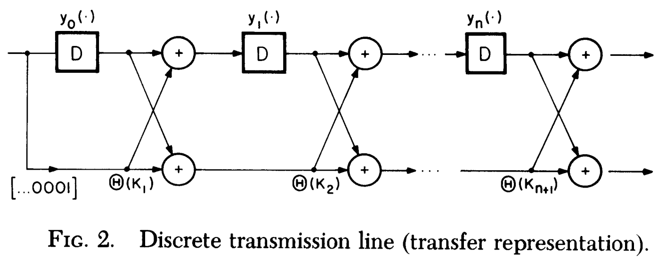

FIG. 2. Discrete transmission line (transfer representation).

Then, we shall show that due to the symmetry of the medium, the matrix

$$\mathbf{Y}_N = \begin{bmatrix} y_0(0) & y_1(0) & y_2(0) & \cdots & y_N(0) \\ 0 & y_1(1) & y_2(1) & \cdots & y_N(1) \\ 0 & 0 & y_2(2) & \cdots & y_N(2) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & y_N(N) \end{bmatrix} \tag{1.10}$$

is an upper Cholesky factor of the inverse of the Toeplitz matrix $\mathbf{T}_N$ defined in (1.1), i.e.,

$$\mathbf{T}_N^{-1} = \mathbf{Y}_N \mathbf{Y}_N^T. \tag{1.11}$$

Moreover, letting

$$Y_N^* = \begin{bmatrix} y_N(N) & y_N(N-1) & y_N(N-2) & \cdots & y_N(0) \end{bmatrix}^T,$$

$$Y_N = \begin{bmatrix} 0 & y_N(0) & y_N(1) & y_N(2) & \cdots & y_N(N-1) \end{bmatrix},$$

we shall show, by using an argument based on signal flow reversal, that we have

$$\mathbf{T}_N^{-1} = L(Y^{*N})L^T(Y^{*N}) - L(Y_N)L^T(Y_N). \tag{1.12}$$

This is a special case of a formula due to Gohberg and Semencul, who derived a similar expression for not necessarily symmetric or positive definite Toeplitz matrices (see e.g. [8, Chapter 4]). Using the fact that a general positive definite matrix of the form (1.2) is always congruent to a Toeplitz matrix via a congruence matrix which is lower triangular, we then show that similar results hold for the inverse of $\mathbf{R}_N$. Explicitly the results are as follows.

Given $\mathbf{R}_N$ of the form (1.2), we have

$$\mathbf{R}_N = L(U_N - V_N)\mathbf{T}_N L^T(U_N - V_N), \tag{1.13}$$

where $\mathbf{T}_N$ is Toeplitz. Then $\mathbf{R}_N^{-1}$ is factored into an upper-lower product, and the factor matrices can be read out as signals on the transmission line when fed with the sequence $\Gamma = [\gamma_0 \ \gamma_1 \ \gamma_2 \ \cdots \ \gamma_N \ldots]$ that defines the lower triangular Toeplitz matrices $L^{-1}(U_N - V_N)$. The Gohberg-Semencul-type for-

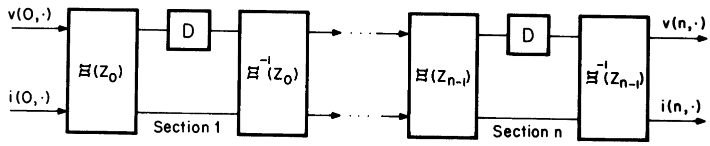

FIG. 3. The voltage-current evolution.

mulas that result are of the form (1.5), where F_N is the convolution of Y_N^* corresponding to $\mathbf{T}_N$ with Γ_N, and G_N is the convolution of Y_N with the same sequence.

2. DISCRETE TRANSMISSION-LINE MODELS

A transmission line with piecewise constant impedance profile propagates impulses of current and voltage. Regarding signals of the form $\sum_{i \in N}\alpha_i\delta(t-i)$ as sequences of numbers defining discrete time series, a transmission line can be modeled as a cascade of elementary sections which are order-1 linear systems (see e.g. [4], [5]). The propagation of the voltage and current sequences along the line is described by the propagation equations (see Figure 3).

$$\begin{bmatrix} V(n+1,i) \\ I(n+1,i) \end{bmatrix} = \Xi^{-1}(Z_n)\begin{bmatrix} \Delta & 0 \\ 0 & 1 \end{bmatrix}\Xi(Z_n)\begin{bmatrix} V(n,i) \\ I(n,i) \end{bmatrix}. \tag{2.1}$$

Here

$$\Xi(Z) = \begin{bmatrix} Z^{-1/2} & Z^{1/2} \\ Z^{-1/2} & -Z^{1/2} \end{bmatrix}, \tag{2.2}$$

with $Z > 0$ the *local impedance*, and Δ stands for the unit-time-delay operator. Transforming the $\{V, I\}$ variables into the so-called "wave variables," defined as

$$\begin{bmatrix} W_R(n,i) \\ W_L(n,i) \end{bmatrix} = \Theta(Z_n)\begin{bmatrix} V(n,i) \\ I(n,i) \end{bmatrix}, \tag{2.3}$$

we obtain the evolution equations

$$\begin{bmatrix} W_R(n+1,i) \\ W_L(n+1,i) \end{bmatrix} = \Theta(k_n)\begin{bmatrix} \Delta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} W_R(n,i) \\ W_L(n,i) \end{bmatrix}. \tag{2.4}$$

Here, the gain matrix $\Theta(k_n)$ is defined as

$$\Theta(k_n) = \Xi(Z_{n+1})\Xi^{-1}(Z_n) = \frac{1}{(1-k_n)^{1/2}}\begin{bmatrix} 1 & -k_n \\ -k_n & 1 \end{bmatrix}, \tag{2.5}$$

where

$$k_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} - Z_n} \tag{2.6}$$

are the local "reflection coefficients". The flow of signals described by (2.4) is the transmission-line model of Figure 2. Wave variables physically propagate in opposite directions: W_R propagates from left to right and W_L propagates from right to left. Therefore, a scattering description is physically more appealing than the transfer description of (2.4). The scattering description of an elementary line section relates the pair $\{W_R(n+1,i), W_L(n,i)\}$ to the incoming waves $\{W_R(n,i), W_L(n+1,i)\}$, and describes the physical interaction between the waves propagating in opposite directions. For the gain matrix we have, by some simple algebra,

$$\begin{aligned} \begin{bmatrix} W_R(n+1,i) \\ W_L(n,i) \end{bmatrix} &= \begin{bmatrix} (1-k_n^2)^{1/2} & -k_n \\ k_n & (1-k_n^2)^{1/2} \end{bmatrix}\begin{bmatrix} W_R(n,i-1) \\ W_L(n+1,i) \end{bmatrix} \\ &= \Sigma(k_n)\begin{bmatrix} W_R(n,i-1) \\ W_L(n+1,i) \end{bmatrix} \end{aligned} \tag{2.7}$$

where $\Sigma(k)$ is the scattering matrix. Note that, as an immediate consequence of losslessness, we have

$$\Theta(k) \text{ is } J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\text{-orthogonal}, \quad \text{i.e.,} \quad \Theta(k)J\Theta^T(k) = J, \tag{2.8}$$

$$\Sigma(k) \text{ is orthogonal}, \quad \text{i.e.,} \quad \Sigma(k)\Sigma^T(k) = I. \tag{2.9}$$

These relations show that at the level of each transmission-line section we have local energy conservation, i.e., the sum of squared input values equals the sum of squared outputs (in the scattering description). Clearly, this local energy-conservation property has further important consequences.

Now, it is clear how the transmission-line model of Figure 1 arises. The lossless cascade system depicted in Figure 1 is parametrized by the sequence of reflection coefficients $\{k_n\}$. It follows from the positivity of the transmission-line impedance profile that

$$|k_n| < 1 \qquad \text{for all } n \tag{2.10}$$

Suppose that the sequence $W_R(0, *) = U = [u_0, u_1, u_2, \ldots, u_N, \ldots]$ is sent into a quiescent transmission line, and the reflected, causal response of the system is $W_L(0, *) = V = [v_0, v_1, v_2, \ldots, v_N, \ldots]$. From Figure 1, it follows immediately that $v_0 = 0$. Furthermore, since the cascade structure modeling the wave propagation on the discrete transmission line is a linear, time-invariant system, we have the following result:

Given an arbitrary sequence $Q = [q_0, q_1, q_2, \ldots, q_N, \ldots]$, the response of the transmission line to the input sequence $W_R(0, \cdot) * Q$ (where $*$ denotes convolution), is the sequence $W_L(0, \cdot) * Q$.

Writing out the convolution in matrix form and considering time lags up to N, we have for any given response pair $\{U, V\}$, in obvious notation,

$$(\text{input})\ U_N^* = L(U_N)\begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_N \end{bmatrix}, \tag{2.11}$$

$$(\text{output})\ V_N^* = L(V_N)\begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_N \end{bmatrix}. \tag{2.12}$$

Incidentally, this shows that for all fixed N, the matrix

$$L(U_N)L^T(U_N) - L(V_N)L^T(V_N) \tag{2.13}$$

is positive definite. Indeed, for any vector $Q_N^* = [q_N, q_{N-1}, \ldots, q_0]^T$, the quantity

$$Q_N^{*T}\left[L(U_N)L^T(U_N)-L(V_N)L^T(V_N)\right]Q_N^* \tag{2.14}$$

is easily seen to measure the difference between the energy sent into the medium by the sequence U^* and the energy that flows out of the medium via V^* up to time N. This however must be a positive quantity, the system being passive.

The Inversion Process; Schur's Algorithm

Consider now the following problem. Suppose we are given a pair of sequences $\{U, V\}$ and we wish to determine whether they are a causal input-response pair for some lossless transmission line, and if this is the case, we also want to find the corresponding sequence of reflection coefficients. Assume that the given sequences do correspond to the response of a transmission line. Then inspection of Figures 1 and 2 shows how to determine the reflection coefficients. Indeed, from the assumed quiescence of the line and by causality of the signal flow it follows that $v_0 = 0$, but also that

$$k_0 = \frac{v_1}{u_0}. \tag{2.15}$$

Once k_0 is determined, we can use $\Theta(k_0)$ to compute the waves $W_R(1,i)$ and $W_L(1,i)$, which form a new causal-response pair for the portion of the line starting at depth 1. Now, however the second reflection coefficient can be determined and the process can be continued recursively. Hence we obtain a simple recursive procedure that determines the sequence $\{k_n\}$ for $n = 0,1,\dots,N-1$ from $\{u_0,u_1,u_2,\dots,u_N\}$ and $\{v_0,v_1,v_2,\dots,v_N\}$. Furthermore, this recursive procedure yields the reflection coefficients up to depth N in $O(N^2)$ computations, since we need $2N+1$ elementary operations per transmission-line section. Note that the reflection-coefficient computations have a nice, nested structure. The algorithm described above is in fact the prototype of a number of results in inverse scattering theory, an inverse scattering problem being the determination of the parameters of a layered medium from data gathered at the boundaries. We shall call the above-described inversion algorithm the Schur procedure, since it implicitly appears in a paper of I. Schur of 1917 [16], dealing with the problem of determining whether a function of the complex variable z, analytic inside the unit disc, is bounded by 1 there. The connection to our problem is the following: if $u_0 = 1$ and $u_n = 0$ for all $n > 0$, the impulse response of the line $[s_0, s_1, s_2, \dots, s_N, \dots]$ is obtained as V. The passivity of the transmission line implies that the function

$$zS_0(z) = zs_1 + z^2 s_2 + z^3 s_3 + z^4 s_4 + \cdots, \tag{2.16}$$

where the s_i's are the lags of the impulse response, is bounded by 1 on the unit disc. Thus $|S_0(z)| < 1$ for $|z| = 1$. Schur proved that any function $S_0(z)$ is bounded on the unit disc iff the sequence of numbers $\{k_n\}$ defined by the recursion

$$S_{n+1}(z) = \frac{1}{z}\frac{S_n(z) - k_n}{1 - k_n S_n(z)}, \qquad k_n = S_n(0), \tag{2.17}$$

obeys $|k_n| < 1$ for all n. Writing the impulse response of the line as

$$\frac{W_L(0,z)}{W_R(0,z)} = zS_0(z) \tag{2.18}$$

and making the recursion (2.17) explicit in terms of the waves W_R and W_L results precisely in the inverse scattering algorithm described above. In fact Schur himself proposed such a "linearized" implementation of his boundedness test. Therefore the classical result of Schur in terms of transmission-line theory is: a complex function in z is bounded on the unit disc if and only if there exists a lossless transmission line having it as its causal reflection function.

We briefly note one more connection to complex-function theory. Suppose we probe the transmission line as follows: we put a perfect reflector boundary at the origin that forces $v_n = u_n$ for $n > 1$ and set $u_0 = 1$. This probing setup is drawn in Figure 4. In this case we have

$$W_R(0,z) = 1 + \tilde{C}(z) = \frac{C(z)+1}{2}, \tag{2.19a}$$

$$W_L(0,z) = \tilde{C}(z) = \frac{C(z)-1}{2}, \tag{2.19b}$$

where $\tilde{C}(z)$ and $C(z)$ are defined accordingly. The connection between the wave transfer function $zS_0(z)$ and the function $C(z)$ is obviously

$$zS_0(z) = \frac{C(z)-1}{C(z)+1}. \tag{2.20}$$

This perfect-reflection experiment corresponds, in the $\{V, I\}$ description, to providing the voltage response $V(0,t)=\delta(t)+\sum_{i=1}^{\infty}c_i\delta(t-i)$ to a forcing current $I(0,t)=\delta(t)$. It is a classical result that if $|S(z)|<1$ on the unit disc then $C(z)$ is a positive real function (or Caratheodory-type function). This implies that the symmetric Toeplitz matrices

$$\mathbf{T}_N=[c_{i-j}]_{i,j=0}^{N}, \qquad c_0=1, \tag{2.21}$$

are positive definite for all N. This result we already know, since it is easy to check that

$$\mathbf{T}_N=L(U_N)L^T(U_N)-L(V_N)L^T(V_N) \tag{2.22}$$

for $U_N=[1\ c_1\ c_2\ c_3\ \cdots\ c_N]^T$ and $V_N=[0\ c_1\ c_2\ c_3\ \cdots\ c_N]^T$.

We remark that the Schur algorithm is in fact an implicit test for positivity of matrices of the form $L(U_N)L^T(U_N)-L(V_N)L^T(V_N)$ and in particular of Toeplitz matrices. The matrix is positive definite provided the sequence of reflection coefficients obtained in the inversion process obeys $|k_n|<1$.

3. ENERGY CONSERVATION AND FAST CAUSAL FACTORIZATION

Suppose we are given two time sequences U and V that form a causal response pair for some discrete, lossless transmission-line model. Equivalently, the matrix

$$\mathbf{R}_N=L(U_N)L^T(U_N)-L(V_N)L^T(V_N) \tag{3.1}$$

is positive definite for all N. Then we can construct a lossless transmission line (via the Schur process) that responds to U_N with V_N. Also, by the linearity of the transmission-line model, if $\{U^1,V^1\}$ and $\{U^2,V^2\}$ are two causal response pairs, the response to $\gamma_1U^1+\gamma_2U^2$ will be $\gamma_1V^1+\gamma_2V^2$. Any portion of the discrete transmission line is a linear system, the state vector at time i, $\mathbf{x}(i)$, being defined as the inputs to the delay elements. The delay elements store their input for one unit of time and then feed into the next section the stored value. The structure of the transmission-line model is such that, by causality of the signal propagation, the sequence of state vectors $\mathbf{x}(i)=[x_0(i)\ x_1(i)\ x_2(i)\ \cdots\ x_N(i)]$ obeys

$$x_j(i)=0 \qquad \text{for} \quad j>i. \tag{3.2}$$

This is obvious, since the right-propagating wavefront, which generates the reflected response, travels at the speed of one section per unit time. The linearity of the model also provides that the state history for a linear combination of inputs is the corresponding linear combination of state vectors. With these preliminatires we can show that the stacked state history yields the "causal-anticausal" or "lower-upper **LU**" decomposition of the matrix $\mathbf{R}_N$. Indeed, it is clear that, by energy conservation, we have to have at all times t that the total energy in the state equals the difference between the energy that went into the system up to time t and the energy that left the system via the output during the same time. Formally, it can be readily proved from local energy conservation [cf. (2.9)] and causality that

$$\sum_{i=0}^{t} u_i^2 - \sum_{i=0}^{t} v_i^2 = \sum_{i=0}^{t} x_i^2(t). \tag{3.3}$$

Define now the row vectors Ψ_t of dimension $2N+2$ as follows:

$$\Psi_t = \begin{bmatrix} u_t & u_{t-1} & \cdots & u_0 & 0 & \cdots & 0 & v_t & v_{t-1} & \cdots & v_0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.4}$$

and then (3.3) shows that

$$\Psi_t \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \Psi_t = \mathbf{x}(t)\mathbf{x}^T(t). \tag{3.5}$$

We can further show that in fact

$$\Psi_t \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \Psi_s^T = \mathbf{x}(t)\mathbf{x}^T(s). \tag{3.6}$$

This follows from the fact that

$$2\langle \Psi_t, \Psi_s \rangle_* = \langle \Psi_t + \Psi_s, \Psi_t + \Psi_s \rangle_* - \langle \Psi_t, \Psi_t \rangle_* - \langle \Psi_s, \Psi_s \rangle_* \tag{3.7}$$

and that, by linearity and time invariance,

$$\langle \Psi_t + \Psi_s, \Psi_t + \Psi_s \rangle_* = \langle x(t) + x(s), x(t) + x(s) \rangle \tag{3.8}$$

In the above expressions $\langle \cdot, \cdot \rangle_*$ denotes the indefinite inner product, as defined by the left-hand side of (3.6), say. Without the asterisk, the usual inner product is invoked. Thus (3.6) is proved, and by stacking (3.5) and (3.6) for $t = 0, 1, \ldots, N$ it follows that

$$\mathbf{T}_N = L(U_N)L^T(U_N) - L(V_N)L^T(V_N) = \mathbf{X}\mathbf{X}^T. \tag{3.9}$$

Therefore, a simple causality argument, together with linearity and energy conservation, shows that a transmission-line model provides the Cholesky factorization of matrices of type $\mathbf{R}_N$. In particular, if some special sets of scattering data, or causal-response pairs, are chosen, we get the factorization of various structured positive definite matrices. For example, response-response pairs of the form $\{[1, c_1, c_2 \cdots c_N], [0, c_1, c_2 \cdots c_N]\}$, corresponding to perfect reflection experiments as discussed in the previous section, yield the factorization of positive definite Toeplitz matrices $\mathbf{T}_N$ as defined by (2.18). This follows from the easily verified identity $\mathbf{T}_N = L(E_N + C_N)L^T(E_N + C_N) - L(C_N)L^T(C_N)$. Furthermore the impulse-response data for the wave-propagation model provide factorizations of matrices of the form $I + L(S)L^T(S)$.

The factorization of matrices of the form given by (1.2) is thus seen to be as easy as the factorization of Toeplitz matrices. An algebraic explanation of this fact follows from the observation, also due to Schur, that $\mathbf{R}_N$ is congruent to a Toeplitz matrix. Indeed, we have the identity

$$L(U)L^T(U) - L(V)L^T(V)$$
$$= L(U-V)\left[\frac{L^{-1}(U-V)L(U+V) + L^T(U+V)L^{-T}(U-V)}{2}\right]L^T(U-V), \tag{3.10}$$

and the matrix inside the square brackets is easily recognized to be Toeplitz. [In the above we have dropped the subscripts, but clearly we are dealing with matrices of size $(N+1)\times(N+1)$.]

The factorization algorithms described above have a computational complexity of $O(N^2)$, the operation count of the general Schur algorithm. We note here that it is *only* due to the particular delay structure of the transmission-line model that we obtain a lower-upper factorization algorithm. If, for example, a completely equivalent model (from the I/O point of view) were chosen with a different location for the delay elements, say on the lower line in Figure 1, the state history would not provide an **LU** factorization. Although it will be true in general that

$$L(U)L^T(U)-L(V)L^T(V)=\mathbf{X}\mathbf{X}^T, \tag{3.11}$$

the matrix **X** will in general be (countably) infinite-dimensional and not lower triangular, as before. The result of causal-anticausal factorization follows from the fact that if the state-history matrix is lower triangular, then we have a nice nesting property: the $(N+1)\times(N+1)$ principal minor of the complete (infinite-dimensional) product $\mathbf{X}\mathbf{X}^T$ is equal to the product of the corresponding minors in **X**.

Matrices of the form $L(U)L^T(U)-L(V)L^T(V)$ are characterized by having "displacement" structure of the form

$$\mathbf{R}_N-Z\mathbf{R}_N Z^T=\begin{bmatrix} U_N & V_N\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}U_N^T\\ V_N^T\end{bmatrix}, \tag{3.12}$$

where Z is the shift matrix having ones in the positions $(i,i-1)$ and zeros everywhere else. It is possible to extend the above methodology to the factorization of matrices that have more complicated displacement structures, with displacement rank higher than two and arbitrary displacement inertia (p,q). The general form of such matrices is

$$\mathbf{R}_N^*=\sum_1^p L(U^i)L^T(U^i)-\sum_1^q L(V^j)L^T(V^j) \tag{3.13}$$

For an account of results concerning matrices with this structure see e.g. [7], [10], [11], [13].

4. TRANSMISSION-LINE TRANSFER-FUNCTION PROPERTIES

In this section we shall derive several properties of transmission-line transfer functions that are needed for the derivation of results concerning factorizations of inverses of structured matrices. The discrete signal propagation model under study is described by the polynomial (or z-transform) domain recursions (see Figure 2)

$$\begin{bmatrix} W_R(n,z) \\ W_L(n,z) \end{bmatrix} = \Theta(k_{n-1}) \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} W_R(n-1,z) \\ W_L(n-1,z) \end{bmatrix}, \qquad (4.1)$$

where

$$\Theta(k) = \frac{1}{(1-k^2)^{1/2}} \begin{bmatrix} 1 & -k \\ -k & 1 \end{bmatrix}.$$

The matrix transfer function $M_N(z)$ relating the signals at the origin to the signals at depth N is therefore given by

$$M_N(z) = \begin{bmatrix} m_{11}(N,z) & m_{21}(N,z) \\ m_{21}(N,z) & m_{22}(N,z) \end{bmatrix} = \prod_{N-1}^{0} \left\{ \Theta(k_i) \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \right\}. \qquad (4.2)$$

The persymmetry of the $\Theta(k)$ matrices, i.e. the fact that

$$\tilde{I}\Theta(k)\tilde{I} = \Theta(k), \qquad \text{where} \quad \tilde{I} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

induces a series of important properties for the transfer-function matrices $M_N(z)$. Clearly, the entries of these matrices are polynomials in z of degree at most n. Also a simple argument based on the symmetry of $\Theta(k)$ provides that

$$m_{22}(N,z) = z^N m_{11}(N,z^{-1}),$$
$$m_{12}(N,z) = z^N m_{21}(N,z^{-1}), \qquad (4.3)$$

i.e. the elements on the diagonals are reversed polynomials. We note once more that these results follow not from the losslessness of the medium but from its symmetry in the transfer representation. In fact any sequence of gain matrices Θ_n of the form

$$\Theta_n = \begin{bmatrix} \alpha_n & \beta_n \\ \beta_n & \alpha_n \end{bmatrix} \qquad (4.4)$$

will induce the same overall properties for $M_N(z)$. By causality of propagation in the scattering domain we know that if $W_R(0,i)=[u_0\ u_1\ u_2\ \cdots\ u_N\ \cdots]$ and $W_L(0,i)=[0\ v_1\ v_2\ \cdots\ v_N\ \cdots]$ are an input-response pair for the transmission line, then at depth N we shall have

$$W_R(N,i)=\begin{cases}0 & \text{for } i<N,\\ \prod_0^{N-1}(1-k_i^2)^{1/2} & \text{for } i=N,\end{cases} \tag{4.5a}$$

$$W_L(N,i)=0 \qquad \text{for } i\leqslant N. \tag{4.5b}$$

Writing out the above results explicitly in the time domain in matrix form, we readily obtain the following basic set of linear equations

$$L(U_N)m_{11}^N+L(V_N)m_{12}^N=\begin{bmatrix}0\\0\\0\\ \vdots\\ \prod_0^{N-1}(1-k_i^2)^{1/2}\end{bmatrix}, \tag{4.6a}$$

$$L(U_N)m_{21}^N+L(V_N)m_{22}^N=\begin{bmatrix}0\\0\\0\\ \vdots\\ 0\end{bmatrix}. \tag{4.6b}$$

These equations simply follow from the fact that the waves at depth N are the convolution of the transfer-function matrix with the data. We have used the following notation: m_{ij}^N represents a column vector stacking the successive coefficients of $m_{ij}(N,z)$ in increasing order (of powers of z), and, as usual, $L(X)$ is a lower triangular Toeplitz matrix with X as its first column. Together with (4.3), which can be rewritten as

$$m_{11}^N=\tilde{I}m_{22}^N \quad \text{and} \quad m_{12}^N=\tilde{I}m_{21}^N, \tag{4.7}$$

these relations can be used to determine the entries of $M_N(z)$ from the scattering data. They are the basic and most general relations for inverse scattering via matrix equations. The classical approach to inverse scattering for transmission-line models is indeed via solutions of matrix equations; however, those are usually derived in the literature for particular types of scattering data. The general approach, which works for arbitrary data sets, is described in [5]. For a survey of classical inverse-scattering results in the discrete case see e.g. [3] or [6].

Considering the set of equations (4.6), we can solve for m_{21}^N in the second equation, and substituting the result into the first one, we obtain

$$\left[L(U_N)-L(V_N)\tilde{I}L^{-1}(U_N)L(V_N)\tilde{I}\right]m_{11}^N=\begin{bmatrix}0\\0\\0\\\vdots\\\prod_0^{N-1}\left(1-k_i^2\right)^{1/2}\end{bmatrix}. \tag{4.8}$$

Also, by using the symmetry relations (4.7) and adding the equations of (4.6) we get

$$\left[L(U_N)+L(V_N)\tilde{I}\right]\left(m_{11}^N+m_{21}^N\right)=\begin{bmatrix}0\\0\\0\\\vdots\\\prod_0^{N-1}\left(1-k_i^2\right)^{1/2}\end{bmatrix}. \tag{4.9}$$

Writing out the above equation for the case when U is a unit impulse sequence $U=[1\ 0\ 0\ \cdots\ 0\ \cdots]$, so that V is the reflection function or impulse response data $V=[0\ s_1\ s_2\ s_3\ \cdots\ s_N\ \cdots]$, results in the Marchenko equation (see [3]). The equations of Gelfand and Levitan invoke the "perfect reflection" case discussed in the previous section. Indeed, if $U=[1\ c_1\ c_2\ c_3\ \cdots\ c_N\ \cdots]$ and $V=[0\ c_1\ c_2\ c_3\ \cdots\ c_N\ \cdots]$ (see Section 3), we obtain an equation of the Gelfand-Levitan type, with a Hankel + Toeplitz kernel (or coefficient matrix). However for the "perfect reflection" case there is another classical inversion equation, due to Krein, having a symmetric Toeplitz kernel. To obtain this equation, note that (4.6) becomes, for $L(U_N) = I + L(C_N)$ and $L(V_N) = L(C_N)$,

$$m_{11}^N + L(C_N)\left(m_{11}^N + m_{12}^N\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \prod_0^{N-1} \left(1 - k_i^2\right)^{1/2} \end{bmatrix}, \tag{4.10a}$$

$$m_{21}^N + L(C_N)\left(m_{21}^N + m_{22}^N\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.10b}$$

But from (4.10) and (4.7) it follows that

$$\left\{I + \left[L(C_N) + \tilde{I}L(C_N)\tilde{I}\right]\right\}\left(m_{11}^N + m_{12}^N\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \prod_0^{N-1} \left(1 - k_i^2\right)^{1/2} \end{bmatrix}. \tag{4.11}$$

The kernel $I + L(C_N) + \tilde{I}L(C_N)\tilde{I} = \mathbf{T}_N$ is symmetric and Toeplitz. This is the Krein equation for discrete inverse scattering. The equations derived above are the basis of the classical scattering algorithms (see e.g. [3] or [5]).

5. FACTORIZATION OF INVERSES OF TOEPLITZ AND RELATED MATRICES

It is known and easy to check that stacking the successive solutions of equations of the form

$$\mathbf{R}_n a_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ a_n^{-1}(n) \end{bmatrix}, \tag{5.1}$$

where $\mathbf{R}_N$ is symmetric and $a_n(i)$ denote the entries of the column vector a_n, yields the upper-lower factorization of $\mathbf{R}_N^{-1}$

$$\mathbf{R}_N^{-1} = \mathbf{U}\mathbf{U}^T, \tag{5.2}$$

where

$$\mathbf{U} = \begin{bmatrix} a_1(1) & a_2(1) & a_3(1) & \cdots & a_N(1) \\ 0 & a_2(2) & a_3(2) & \cdots & a_N(2) \\ 0 & 0 & a_3(3) & \cdots & a_N(3) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_N(N) \end{bmatrix}. \tag{5.3}$$

In the previous section we have shown that causality of signal propagation on the transmission line, together with some simple symmetry considerations, leads to equations of the type (5.1) for various combinations of vectors that describe the forward transfer function for portions of transmission line of increasing size. Indeed, recalling Equation (4.8), we realize that finding m_{11}^n for $n = 0, 1, 2, \ldots, N$ and stacking the solutions into a $(N+1)\times(N+1)$ matrix provides the upper-lower factorization of

$$\left[L(U_N) - L(V_N)\tilde{I}L^{-1}(U_N)L(V_N)\tilde{I}\right]^{-1} \tag{5.4}$$

Similarly, considering equations of the form (4.9), we have that the vectors $(m_{11}^n + m_{21}^n)$, for $n = 0, 1, 2, \ldots N$, provide by stacking the upper-lower factorization of

$$\left[L(U_N) + L(V_N)\tilde{I}\right]^{-1}, \tag{5.5}$$

i.e. of the inverse of a Toeplitz + Hankel matrix. When writing out the above results for the particular impulse response and perfect reflection data sets, we see that a triangular factorization always underlies the classical inverse-scattering methods. The most interesting factorization is the one related to the Krein system of equations. Equation (4.11) shows that a matrix stacking the vectors $y_n = m_{11}^n + m_{12}^n$, for increasing values of n (and suitably continued with zeros), yields the upper triangular factor of the inverse of the positive definite Toeplitz matrix $\mathbf{T}_N$. Furthermore, for the lossless case of Equation (4.11), we have the nonzero term at the right given by

$$W_R(n,n)=\prod_{i=0}^{n-1}\left(1-k_i^2\right)^{1/2}, \tag{5.6}$$

and it turns out that (see Figure 2)

$$y_n(n)=m_{11}^n(n)+m_{12}^n(n)=\prod_{0}^{n-1}\left(1-k_i^2\right)^{-1/2}. \tag{5.7}$$

Therefore the remarks associated with (5.1) directly apply, and we can say that the symmetric, positive definite Toeplitz matrix $\mathbf{T}_N^{-1}$ is factored as

$$\mathbf{T}_N^{-1}=\mathbf{Y}\mathbf{Y}^T, \tag{5.8}$$

where

$$\mathbf{Y}=\begin{bmatrix} y_0(0) & y_1(0) & y_2(0) & \cdots & y_N(0) \\ 0 & y_1(1) & y_2(1) & \cdots & y_N(1) \\ 0 & 0 & y_2(2) & \cdots & y_N(2) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & y_N(N) \end{bmatrix}. \tag{5.9}$$

A natural question that now arises is: how can we obtain the polynomials $y(n,z)=m_{11}(n,z)+m_{12}(n,z)$, or the columns of the upper triangular factor $\mathbf{Y}$, by using the structure of the transmission line identified by the scattering data (in this case the perfect-reflection scattering data)? The answer is that we can obtain $y(n,z)$ and its reverse polynomial by forming

$$\begin{bmatrix} m_{11}(n,z) & m_{12}(n,z) \\ m_{21}(n,z) & m_{22}(n,z) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} y(n,z) \\ z^n y(n,z^{-1}) \end{bmatrix}. \tag{5.10}$$

Therefore the successive columns of the upper triangular factor of $\mathbf{T}_N^{-1}$ can be obtained by feeding the forward-propagation structure of Figure 2 with the signals $W_R(0,z)=W_L(0,z)=1$. Considering the forward-propagation model, it is therefore clear that the time history of the ith state (input to the ith delay) is the vector y_i. This result should be compared with the corresponding one of Section 3 (see Figure 4), which shows that the state of the

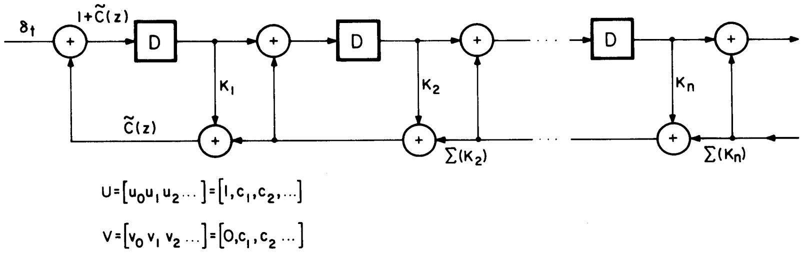

FIG. 4. The reflection experiment.

feedback line driven by the data $U=[1\ c_1\ c_2\ \cdots]$ and $V=[0\ c_1\ c_2\ \cdots]$ determines the lower triangular factor of $\mathbf{T}_N$. The energy-conservation proof of that result used the identity

$$\mathbf{T}_N=\mathbf{X}\mathbf{X}^T=L(E_N+C_N)L(E_N+C_N)-L(C_N)L^T(C_N). \tag{5.11}$$

This suggests that we might find a similar formula for $\mathbf{T}_N^{-1}$ by again appealing to energy-conservation arguments. To carry this out, note that, by reversing the direction of flow in the lower line of Figure 2, which we have just used, we get the equivalent structure of Figure 5. If this structure is now fed from the right with the input $z^n y(n, z^{-1})$, it is immediate to realize that the state history remains the same as in the forward structure of Figure 2. The structure is however lossless. Applying energy-conservation arguments as in Section 3 [Equation (3.10)] shows that the state history on the transmission line provides the factorization of the matrix

$$L(Y_N^*)L^T(Y_N^*)-L(Y_N)L^T(Y_N) \tag{5.12}$$

where Y^* denotes the reversed vector and $Y_N=[0\ y_N(0)\ \cdots\ y_N(N-1)]^T$.

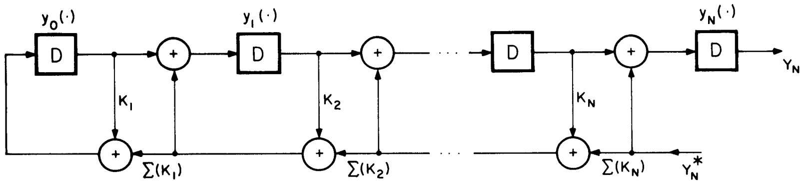

FIG. 5. Proof of the Gohberg-Semencul formula.

But the state history is in this case mapped by $\mathbf{Y}$, and therefore we get

$$\mathbf{Y}\mathbf{Y}^T = \mathbf{T}_N = L(Y_N^*)L^T(Y_N^*) - L(Y_N)L^T(Y_N), \tag{5.13}$$

which is a particular case of the striking formula of Gohberg and Semencul for the inverse of a Toeplitz matrix.

Incidentally, this also proves that if $\mathbf{T}_N$ is a symmetric positive definite Toeplitz matrix with reflection coefficient sequence $\{k_0, k_1, k_2, \dots, k_{N-1}\}$, then its inverse $\mathbf{T}_N^{-1}$ has similar displacement structure and the corresponding reflection coefficients are $\{k_{N-1}, k_{N-2}, \dots, k_0\}$.

6. INVERSION OF GENERAL STRUCTURED MATRICES

In the previous section we have rederived a number of well-known results for inverses of Toeplitz matrices via transmission-line arguments. The special structure of the arguments used to obtain the Krein equation (4.11) that was the starting point of these arguments would not seem to extend to general kernels of the form $L(U)L^T(U) - L(V)L^T(V)$, for which we have obtained direct factorization results in Section 3. However, the congruence reduction of such matrices to Toeplitz form via a lower triangular Toeplitz matrix can be effectively used to obtain the appropriate generalizations of the results for $\mathbf{T}_N^{-1}$. For convenience we shall first restate (and in fact rederive) the congruence identity here.

Given a matrix $\mathbf{R}_N$ of the form (1.2), we wish to determine a congruence matrix Λ_N such that

$$L(U_N)L^T(U_N) - L(V_N)L^T(V_N) = \Lambda_N \mathbf{T}_N \Lambda_N^T. \tag{6.1}$$

Recalling that a symmetric Toeplitz matrix can always be represented as

$$\mathbf{T}_N = L(E_N + C_N)L^T(E_N + C_N) - L(C_N)L^T(C_N), \tag{6.2}$$

we immediately have for Λ_N

$$\Lambda_N L(E_N + C_N) = L(U_N) \quad \text{and} \quad \Lambda_N L(C_N) + L(V_N), \tag{6.3}$$

which determines the congruence matrix to be

$$\Lambda_N = L(U_N - V_N). \tag{6.4}$$

Therefore, if the factorization of $\mathbf{T}_N^{-1}$ is available, the corresponding factorization of $\mathbf{R}_N^{-1}$ will be

$$\mathbf{R}_N^{-1} = L^{-T}(U_N - V_N)\mathbf{Y}\mathbf{Y}^T L^{-1}(U_N - V_N) = \mathbf{UL} \tag{6.5}$$

(**U** upper, **L** lower), i.e., the lower triangular factor of the general structure matrix is $\mathbf{Y}^T L^{-1}(U_N - V_N)$. Finding the inverse of a lower triangular Toeplitz matrix is an easy and efficient computation of complexity $O(N \log N)$; however, the multiplication of a general lower triangular matrix with the Toeplitz matrix $L^{-1}(U_N - V_N)$ requires at least $O(N^2 \log N)$ computations. But the columns of $\mathbf{Y}_N$ are certainly not arbitrary; we can use again the insight provided by the transmission-line model to show that determining the factor of $\mathbf{R}_N^{-1}$ is inherently an $O(N^2)$ computation. To do so, note that the kth row of $\mathbf{Y}_N L^{-1}(U_N - V_N)$ is obtained by passing the sequence $[0, 0, \ldots, 0, y_k(k), y_k(k-1), \ldots, y_k(0)]$ through a linear time-invariant filter with impulse response $\gamma_0, \gamma_1, \gamma_3, \ldots, \gamma_N, \ldots$, where the sequence Γ is defined by

$$L^{-1}(U_N - V_N) = L(\Gamma_N). \tag{6.6}$$

Alternatively, we can regard the resulting row as obtained by passing the sequence Γ through a filter with impulse response $[0, 0, \ldots, y_k(k), y_k(k-1), \ldots, y_k(0)]$. But we know that such a filter is readily available for us. Indeed, the feedforward structure of Figure 2 provides a nested realization of such transfer functions, the output being taken from the lower line. Therefore, applying the sequence $\gamma_0, \gamma_1, \ldots$ as an input to the system of Figure 2, we can read out the rows of the factor of $\mathbf{R}_N^{-1}$ as the first $k+1$ lags of the lower line outputs at stage k. In conclusion, due to the built-in transmission-line structure of the Toeplitz factorization process we obtain fast factorizations of general $L(U)L^T(U) - L(V)L^T(V)$ matrices. Similar results hold for the general displacement structure matrices, as demonstrated in [7], [10]–[13].

Gohberg-Semencul Formulas

We saw that the inverse of $\mathbf{T}_N^{-1}$ can be expressed as

$$\mathbf{T}_N^{-1} = L(Y_N^*)L^T(Y_N^*) - L(Y_N)L^T(Y_N). \tag{6.7}$$

Next we shall derive a similar formula for the inverse of $\mathbf{R}_N$. We saw that it follows by congruence that

$$\mathbf{R}_N^{-1} = L^{-T}(U_N - V_N)\mathbf{Y}\mathbf{Y}^T L^{-1}(U_N - V_N) = \mathbf{UL} \tag{6.8}$$

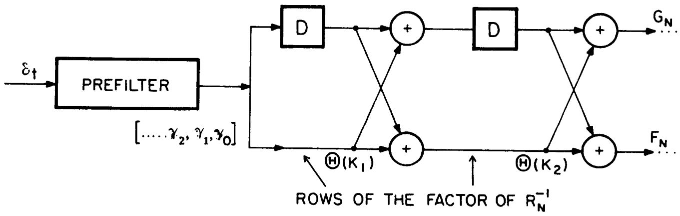

FIG. 6. Factorization of R_N^{-1}.

(**U** upper, **L** lower). Using the persymmetry property of $\mathbf{T}_N$ and $\mathbf{T}_N^{-1}$, i.e. that

$$\tilde{I}\mathbf{T}_N\tilde{I} = \mathbf{T}_N \quad \text{and} \quad \tilde{I}\mathbf{T}_N^{-1}\tilde{I} = \mathbf{T}_N^{-1}, \tag{6.9}$$

we obtain that

$$\tilde{I}\mathbf{R}_N^{-1}\tilde{I} = \tilde{I}L^{-T}(U_N - V_N)\tilde{I}\mathbf{T}_N^{-1}\tilde{I}L^{-1}(U_N - V_N)\tilde{I}. \tag{6.10}$$

From (6.10) and the Gohberg-Semencul formula for $\mathbf{T}_N^{-1}$ it follows that

$$\tilde{I}\mathbf{R}_N^{-1}\tilde{I} = L(F_N)L^T(F_N) - L(G_N)L^T(G_N), \tag{6.11}$$

where

$$F_N = [\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_N] * Y_N^* \tag{6.12}$$

and

$$G_N = [\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_N] * Y_N. \tag{6.13}$$

Note that F_N and G_N are obtained by reading out the output of the transmission line structure of Figure 2 to the "prefilter input sequence" defined by $[\gamma_0, \gamma_1, \dots]$, determined by the matrices $L^{-1}(U_N - V_N)$ (see Figure 6).

7. CONCLUDING REMARKS

In this paper we have presented a rather comprehensive account of results concerning fast factorization of matrices having a certain displacement structure. Toeplitz matrices, for which such results are more frequently encountered in the literature, are just particular cases of a general class of matrices for which fast algorithms can be derived. The analysis of signal propagation in a linear time-invariant system modeling a discrete transmission-line structure proved very useful in deriving, unifying, and summarizing the factorization results. Also it provides a direct route to generalizations of these factorization results to matrices with more complex displacement structures. Such generalizations are discussed in the recent paper of Lev-Ari and Kailath [10] and in many of the references cited therein.

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