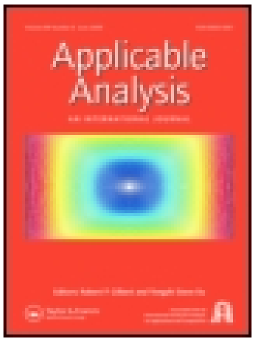

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Epi-convergence of Discrete Elastica

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Curves that pass through specified locations with specified orientations and minimize an energy functional are called elastica. While physical splines readily assume minimal energy configurations, finding the numerical solutions of variational problems involving integrals of nonlinear functions of the curvature remains quite a formidable challenge. Approximate solutions of such problems yield satisfactory results and the computer-aided design field relies heavily on polynomial or rational curve designs. In this paper we discuss a method for discretizing the problem of nonlinear spline design, an alternative to the more traditional approach of discretizing the differential equations that solve the variational problems involved. We show that discretizing the energy functionals (i.e. considering polygonal approximations of the curves and finding the ones that minimize their "energy" defined directly in terms of turn angles and segment length) is an approach that is simpler and leads to solutions that, in the limit of very small segment lengths, converge to the optimal continuous solutions.

Keywords: Curve; Numerical methods; Approximation; Epi-convergence; Variational

AMS: 65D17; 65K10

1 INTRODUCTION

Before computers became the dominant tools in the design of industrial products, elastic rods called splines were widely used in drafting and designing the shapes of smooth curved objects.

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Physically, an elastic rod confined to obey some constraints like passing through a set of points at given directions, will settle into a "minimal energy configuration". The energy of such a bent, elastic rod is a functional of its curvature profile, hence the problem of smooth constrained interpolation curve design naturally leads to variational optimizations involving functionals of curvature, see e.g. [1–3].

Since, in general, solving variational problems with cost functionals dependent on curvature leads to nonlinear differential boundary value problems, the CAD community resorted to approximations that stimulated an active and very interesting and successful area of cubic (or general polynomial) or rational spline curve design, see e.g. [4–6]. The original nonlinear spline design problem continued to be of interest, however, to mathematicians and CAD people alike. The nonlinear problems are, after all, modeling the "real" elastic rods that have led to several wonderful designs in the past, and are mathematically very beautiful and challenging. Recently, the topic of smooth interpolation curves has also arisen in several other areas of endeavor like robotics and computer vision [7–12]. This paper deals with the topic of discretization in conjunction with nonlinear design. When the variational problems of elastica are to be solved to yield usable curve designs the issue of discretization necessarily arises: the computer will have to generate a sequence of points or some other discrete representation of the nonlinear spline. The traditional approach to discretization was the following: reduce the variational problem to a (nonlinear) differential equation and use the arsenal of numerical methods provided by numerical analysis to solve the differential boundary value problems necessary for curve designs. While this approach is indeed natural and yields a wealth of practical nonlinear spline design methods, (see e.g. [13]) another natural approach to discretization seems to have been neglected. The alternative approach that we shall mathematically analyze in this paper proposes the discretization of the spline design problem itself. Instead of aiming at producing smooth splines, this approach seeks polygonal approximations (for the nonlinear splines) directly. The desired curve is therefore a piecewise straight curve with small segments that minimizes an energy functional defined in terms of the segment length and vertex angles of the polygonal spline approximation. This approach was numerically tested for various types of energy functionals in [14] and produced pleasing results. However, if we also want to claim that "discrete elastica" designs are a viable procedure to solve (i.e. discretize) continuous nonlinear spline design problems, we must prove mathematically the convergence of these polygonal curves to their continuous versions. Note that what we are doing is discretizing the problem at the outset by defining energy functionals over polygonal curves and then proving that correct choices of such energy functionals will yield discrete elastica that converge, in the limit of an infinite number segments of infinitesimal length, to the optimal continuous curves that minimize a "limit" energy functional defined over the class of continuous and smooth curves.

In this paper we use the theory of epi-convergence, as developed by DeGiorgi and his associates [15] under the name Γ-convergence, and independently by Attouch [16], to provide such convergence results. To understand the aims and issues that we want to address let us start with a trivial example from the calculus of variations.

Example 1 Suppose we are searching for the function $f(x)$ so that over $[0, 1]$ we minimize

$$\int_0^1 [f'(x)]^2 dx$$

and $f(0) = a$ and $f(1) = b$. This is a very simple problem in the calculus of variations and the Euler equation provides, see e.g. [17, p. 37],

$$\frac{d}{dx} \cdot [2f'] = 0 \quad \text{or} \quad f''(x) = 0.$$

Fortunately, we can solve this equation explicitly to get $f(x) = \alpha x + \beta$ and determine (α, β) that satisfy $f(0) = a$, $f(1) = b$; i.e. $\beta = a$, $\alpha = (b - a)$. If we were not so fortunate in getting an Euler equation that is explicitly solvable we would have had to numerically solve for f satisfying $f''(x) = 0$ and the boundary conditions; i.e. we would have had to discretize $(d^2/dx^2)f(x) = 0$, and this would have led us to determining a function $f(x)$ defined, for example, at $x_0 = 0$, $x_1 = \Delta$, $x_2 = 2\Delta, \ldots x_N = N\Delta = 1$ obeying

$$\frac{1}{\Delta}\left[\left(\frac{f(x_{n+1}) - f(x_n)}{\Delta}\right) - \left(\frac{f(x_n) - f(x_{n-1})}{\Delta}\right)\right] = 0$$

or

$$f(x_{n+1}) - 2f(x_n) + f(x_{n-1}) = 0$$

with $f(x_0) = a$ and $f(x_N) = b$; yielding $\tilde{f}(x_i) = a + (b - a) \cdot i\Delta$ as the approximate solution. This happens to be an *exact* sampling of the true solution ($f(x) = a + (b - a)x$) and, were we not so fortunate as to have the true solution for comparison, we would have had to worry about the quality of the approximate solution $\tilde{f}$ and its behavior as $\Delta \to 0$ ($N \nearrow \infty$). This is, in fact, the main concern of classical numerical analysis.

The method outlined above is the traditional approach to solving the variational problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\int_0^1 (f'(x))^2 dx \\ \text{subject to} & f(0) = a, \quad f(1) = b. \end{array} \tag{1}$$

An alternative approach, the subject of this paper, is to discretize the problem itself. We could have said: let us not look for solutions of the problem in the space of all smooth functions $f(x)$ over $[0, 1]$ but rather for solutions of a discretized problem in a restricted space of functions over $[0, 1]$ parameterized by N discrete values denoted by $\phi(0)$, $\phi(1), \ldots, \phi(N - 1)$. For example, we could have considered the functions that are piecewise constant over the intervals $[0, \Delta]$, $[\Delta, 2\Delta], \ldots, [(N - 1)\Delta, N\Delta]$, where $N\Delta = 1$ (as before), and let $\phi(i) = f^D(i\Delta)$, be the parameters for optimization. Once we decide to proceed in this way we must define a new cost functional that somehow mimics $\int_0^1 (f'(x))^2$ over the space of functions under consideration. A rather straightforward proposal is of course

$$\sum_{i=1}^{N} (\phi(i) - \phi(i - 1))^2$$

and now we can pose the discrete variational problem corresponding to (1) as follows

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{N} (\phi(i) - \phi(i - 1))^2 \\ \text{subject to} & \phi(0) = a, \quad \phi(N) = b \end{array}$$

This problem is readily solved as follows

$$\left\{\frac{\partial}{\partial\phi(i)}\left[\sum_{j=1}^{N}(\phi(j)-\phi(j-1))^2\right]=0\right\} \quad i=1,2,\ldots,N-1$$

or

$$\{[\phi(i)-\phi(i-1)]-[\phi(i+1)-\phi(i)]\}=0 \quad \text{for } i=1,\ldots,(N-1)$$

Hence we have

$$\{\phi(i+1)-2\phi(i)+\phi(i-1)=0\} \quad \text{for } i=1,2,\ldots,(N-1)$$

with $\phi(0)=a$ and $\phi(N)=b$ yielding

$$\phi(i)=a+(b-a)\frac{i}{N}$$

as the optimal solution.

In this simple case the solution of the discrete optimization problem is "identical" in some sense to the discrete solution of the continuous problem. When we cannot exhibit explicit expressions for the solutions of the discretized continuous problem and the corresponding discrete problem we must be concerned about the nature and quality of the approximation.

Example 2 Suppose next we are trying to solve the following variational problem:

$$\text{minimize} \int_0^1 [f'(x)]^2 w(x)dx$$

subject to $f(0)=a$ and $f(1)=b$, where $w(x)$ is a given smooth positive function. In this case the Euler equation is

$$\frac{d}{dx}F_{y'}(x,y')=0$$

where $F(x, y') = w(x)(y')^2$, and $(y \equiv f)$. So:

$$\frac{d}{dx} w(x) 2y'(x) = 0 \Rightarrow \frac{d}{dx} w(x) f'(x) = 0$$

hence

$$f'(x) w(x) = \text{ const.}$$

or

$$w(x) f''(x) + w'(x) f'(x) = 0. \tag{2}$$

The solution of this equation is

$$f^{\text{opt}}(x) = \int_0^x \frac{c}{w(\xi)} d\xi + f(0)$$

where $f(0) = a$ and $\int_0^1 (c/w(\xi)) d\xi + a = b$ determines the constant c. If we have to solve numerically $f'(x) = c/w(x)$ we would proceed via

$$\frac{f^{\text{ND}}(x_{n+1}) - f^{\text{ND}}(x_n)}{\Delta} = \frac{c}{\tilde{w}(x_n)}$$

where $\tilde{w}(x_n) = w(x_n)$ or $(w(x_n) + w(x_{n+1}))/2$ etc. (the superscript ND indicates that f arises from a non-discrete functional), yielding

$$f^{\text{ND}}(x_{n+1}) = f^{\text{ND}}(x_n) + \frac{\Delta c}{\tilde{w}(x_n)} \tag{3}$$

with $f^{\text{ND}}(x_0) = a$ and $f^{\text{ND}}(x_N) = b$. The relation between the solution provided by (3) above and the "true" discrete values at the sampling points now depends on the behavior of $w(x)$ and the choice of the approximation $\tilde{w}(x_n)$. For well behaved $w(x)$ it would be easy to show that a variety of numerical schemes will yield that $f^{\text{ND}}(x_i) \to f^{\text{opt}}(x)$ in various senses.

Let us next analyze the corresponding discretized problem. Assuming we deal with piecewise linear functions between the samples at $0, \Delta, 2\Delta, \ldots, N\Delta = 1$ of the discrete functions $f^{\text{D}}(x)$, [i.e. $f^{\text{D}}(i\Delta) = \phi(i)$] (the superscript D indicates that f arises from a discrete functional), we can now postulate the discrete optimization problem as follows

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^{N} [\phi(i) - \phi(i-1)]^2 \tilde{w}(i\Delta) \\ &\text{subject to} \quad \phi(0) = a \quad \text{and} \quad \phi(N) = b. \end{aligned}$$

Here

$$\left\{ \frac{\partial}{\partial \phi(i)} \left[\sum_{j=1}^{w} [\phi(j) - \phi(j-1)]^2 \tilde{w}(j\Delta) \right] = 0 \right\}_{i=1,2,\ldots,N-1}$$

yields:

$$[\phi(i) - \phi(i-1)]\tilde{w}(i\Delta) - [\phi(i+1) - \phi(i)]\tilde{w}((i+1)\Delta) = 0$$

or

$$\tilde{w}((i+1)\Delta)\phi(i+1) - \phi(i)[\tilde{w}(i\Delta) + \tilde{w}((i+1)\Delta)] - \tilde{w}(i\Delta)\phi(i-1) = 0$$

This equation is equivalent to

$$\begin{aligned} &[\phi(i+1) - 2\phi(i) + \phi(i-1)]\tilde{w}(i\Delta) \\ &\quad - [\phi(i+1) - \phi(i)]\tilde{w}(i\Delta) + [\phi(i+1) - \phi(i)]\tilde{w}((i+1)\Delta) = 0 \end{aligned}$$

or

$$\begin{aligned} &[\phi(i+1) - 2\phi(i) + \phi(i-1)]\tilde{w}(i\Delta) \\ &\quad + [\tilde{w}((i+1)\Delta) - \tilde{w}(i\Delta)]\,[\phi(i+1) - \phi(i)] = 0 \end{aligned}$$

an equation that can be recognized as the discretization of (2).

Here again the question arises whether the discretizations of the problem yield solutions that approach, as $N \nearrow \infty$, the continuous solution of the original variational problem. This question could, here too, be addressed rather directly, yielding conditions on the relations between the discrete cost functions and the continuous one to ensure the convergence of the solutions to the continuous optimizing function. The theory of Γ-convergence, to be discussed and used in the sequel provides general conditions under which discrete functionals (cost functions) of discretized functions yield in the limit of $N \nearrow \infty$ minimizers of desired continuous functionals of continuous and smooth functions.

Example 3 Let us consider extending Example 2 so that the Euler–Lagrange approach becomes problematic. Suppose that the f of Example 2 is to serve as a boundary separating the box $\{(x,y) : x \in [0,1], y \in [-A, A]\}$ (A sufficiently large) into two regions. Suppose further that there is a function $g(x,y)$ which is to be smoothed, by averaging, in both regions independently and that the boundary is chosen to minimize the discrepancy between g and its piecewise smoothed versions.[1] Thus, we seek to minimize:

$$\int_{y>f(x)} \int_0^1 (g(x,y) - c_1)^2 dx\, dy + \int_{y<f(x)} \int_0^1 (g(x,y) - c_2)^2 dx\, dy \\ + \int_0^1 (f'(x))^2 dx, \quad \text{subject to } f(0) = a, \quad f(1) = b. \tag{4}$$

(Here c_1 and c_2 are constants to be optimized: they will be averages of g over the corresponding regions.) In this case it is possible to derive Euler–Lagrange equations for f, but they are complicated and non-local so that direct numerical solution is not obvious.

If, alternatively, we discretize the box $[0,1] \times [-A, A]$ then it is clear how to discretize the functional above. The Γ-convergence theory allows one to conclude convergence of solutions of the discretized problem to solutions of the continuous problem from the corresponding convergence of solutions in Example 2. The reason for this is that the extra terms introduced depend "continuously" on f.

Example 4 This example is the topic of planar curve design, and introduces some of the issues that will be addressed in the main part of our paper. An intrinsic description of a planar curve is the direction angle versus arc length, or $\psi(s)$, description. Suppose we want to design a planar curve that starts at the origin $(0,0)$, ends at $(1,0)$, is of given length L, with $\psi(0) = \psi_0$ and $\psi(L) = \psi_f$ (i.e. preset initial and final directions) and minimizes the elastic energy measured

[1]This simple example is based on the variational formulation of edge detection in images [19].

by the integrated squared curvature $k(s) = (d/ds)\psi(s)$ over $[0, L]$: $\mathbf{C}(\psi) = \int_0^L [(d/ds)\psi(s)]^2 ds$. This problem leads to the following Euler–Lagrange differential equation:

$$2\frac{d^2}{ds^2}\psi(s) = -\lambda_1 \sin\psi(s) + \lambda_2 \cos\psi(s) \tag{5}$$

subject to $\psi(0) = \psi_0$ and $\psi(L) = \psi_f$, where λ_1 and λ_2 are Lagrange multipliers. A variety of numerical methods could be used to solve (5) to determine $\psi(s)$ that obey the boundary conditions, i.e.,

$$\int_0^L \cos\psi(s)ds = \Delta x = 1$$
$$\int_0^L \sin\psi(s)ds = \Delta y = 0$$

and $\psi_0 = \psi(0)$; $\psi_f = \psi(L)$. However, here too, we could proceed by discretizing the problem itself. We could consider approximately the curve by an n-link polygonal chain having, say, equal $\ell = (1/n)L$-length straight segments. In this setting we could pose the problem of determining the poly-line that minimizes a discrete version of the elastic energy. The curvature in this energy functional could be measured by the turn angle from one chain link to the next. Suppose the links are oriented at the angles $\psi_0, \psi_1, \psi_2, \ldots, \psi_{N-1} = \psi_f$. Then $k_i = (\psi_{i+1} - \psi_i)/\ell$ could serve as a discrete curvature and we could pose the problem of minimizing

$$\sum_{i=1}^{n-2} \ell k_i^2 = \left(\frac{1}{\ell}\right) \sum_{i=0}^{n-2} (\psi_{i+1} - \psi_i)^2$$

subject to

$$\sum_{i=0}^{n-1} \ell \sin\psi_i = 0$$
$$\sum_{i=0}^{n-1} \ell \cos\psi_i = 1$$

with ψ_0 and $\psi_f = \psi_{N-1}$ predetermined. (See Fig. 1.)

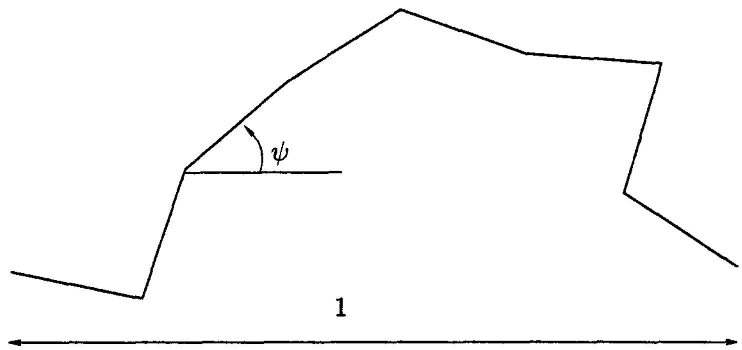

FIGURE 1 Polygonal curve.

This problem leads to a system of nonlinear equations for ψ_1, $\psi_2, \ldots, \psi_{n-2}$ as opposed to the differential equation (5) two point boundary value problem for the continuous problem. Indeed using the Lagrange function

$$J(\psi_1, \ldots, \psi_{n-1}) := \sum_{i=0}^{n-2} \frac{(\psi_{i+1} - \psi_i)^2}{\ell} \lambda_1 \sum_{i=0}^{n-1} \ell \sin \psi_i + \lambda_2 \sum_{i=0}^{n-1} \ell \cos \psi_i$$

and differentiating with respect to $\psi_1, \psi_2, \ldots, \psi_{n-2}$ we obtain:

$$\begin{gathered}\left\{2(\psi_{i+1} - 2\psi_i + \psi_{i-1}) - \lambda_1 \ell^2 \cos \psi_i + \lambda_2 \ell^2 \sin \psi_i = 0\right\} \\ i = 1, 2, \ldots, n-2\end{gathered}$$

with ψ_0 and $\psi_{n-1} = \psi_f$. This system of equations can be solved by a variety of methods in terms of the Lagrange multipliers λ_1, λ_2 that must be set to meet the required start-point/endpoint conditions, see e.g. [11,14]. The question that arises in this context is the following: if we increase n, we would like to obtain a sequence of polygonal approximations to the solution of the original continuous problem. However, note that we have not discretized the solution of the continuous problem but rather we have discretized the problem itself. As such, the sequence of solutions that we obtain are polygonal lines (non-smooth by definition) that should, in some sense, converge to the solutions of a continuous problem.

This paper presents and discusses the theory of Γ-convergence of such approximations, obtained by discretizing the problems at the outset, to the solutions of continuous variational problems involving several types of cost functions that are functionals of curvature for planar curves.

The curvature dependent functionals that were proposed and discussed in the paper Discrete Elastica, [14] and for which convergence results will be proved herein are the following:

(1) $\int_0^L |k|^{2\alpha} ds$ – classical elastica;
(2) $L \cdot \int_0^L |k|^2 ds$ – similarity invariant elastica;
(3) $\int_0^L (k/K_{\max})^{2\alpha} ds$ – elastica with hard limits on turn.

Note that (1) and (3) are equivalent. The distinct interest in (3) arises by considering allowing α to tend to infinity. The limiting functional is simply a hard limit on curvature. From the perspective of Γ-convergence theory, functional (2) is a continuous perturbation of functional (1). Thus, convergence results for (2) will follow easily from those for (1). Hence, in our technical development, we will focus on (1).

Consider a polygonal curve consisting of segments $\vec{\ell}_1, \ldots, \vec{\ell}_n$ with lengths $\ell_1, \ldots, \ell_n$. (To simplify the presentation we will always assume that the initial point of a polygonal curve is the origin and we will denote by $[\vec{\ell}_1, \ldots, \vec{\ell}_n]$ the polygonal curve whose kth vertex is given by $\sum_{i=1}^{k-1} \vec{\ell}_i$.) Corresponding to the functional $\int_0^L |k|^{2\alpha} ds$, we consider discrete functionals of the form

$$\sum_{i=1}^{n-1} G(\vec{\ell}_i, \vec{\ell}_{i+1})$$

defined over n-segment polygons.

As we shall see using the general Γ-convergence approach, we can prove that various such discrete functionals approximate the continuous one in a rigorous sense. The main implication of this approximation is that minimizers of the discrete problems converge to minimizers of the continuous problems as n tends to infinity.

2 EPI-CONVERGENCE FOR ELASTICA

As our examples indicate, many optimization problems are amenable to variational analysis leading to differential equations satisfied by optimal solutions and Euler's elastica is a classical example. As discussed in the introduction, one could attempt to compute a solution to the differential equation directly. In many other situations it is very difficult to obtain Euler–Lagrange equations much less solve them. An alternative approach, better suited to such situations would be to discretize the functional to be minimized and then to minimize the discrete function directly using relaxation methods or other numerical methods. The theory of Γ-convergence then allows one to rigorously prove approximation properties of the solutions obtained thereby.

In this paper we focus on functionals defined over curves. We are interested in approximating functionals that depend on norms of the curvature of the curve, e.g.,

$$F_\alpha(\gamma) := \int_0^{L(\gamma)} |\kappa(s)|^\alpha \, ds$$

where $1 < \alpha < \infty$. The case $\alpha = 1$, $F_1(\gamma)$, is the total absolute curvature of γ. It has a well known analog for polygonal curves: the sum of the exterior angles. We are primarily interested in the case $\alpha > 1$ for which provably good approximations are not well known.

The functionals we consider are often introduced in applications as regularization terms. Other terms depending on the curve usually do so in a lower-order way, e.g., they depend continuously on the curve. A Γ-convergence result on the regularization term is easily extended since Γ-convergence is stable under continuous perturbations.

Given a functional F defined, essentially, over smooth curves we will define functionals K_n, essentially, over polygonal curves in such a way that

$$K_n \xrightarrow{\Gamma} F$$

where here $\xrightarrow{\Gamma}$ denotes Γ-convergence. The word 'essentially' appears here because the Γ-convergence theory requires that all curves live in one ambient metric space, so functionals K_n and F must be defined over the entire space. This is accomplished in a trivial way: the functionals are set to $+\infty$ on curves over which they are not intrinsically well defined.

The general structure of Γ-convergence theory is the following. Let X be a separable metric space and let F_n, $n = 1, 2, \ldots$ and F be functionals defined over X. We say F_n Γ-converges to F if

(L) $\forall x \in X : x_n \to x \Rightarrow \liminf_{n\to\infty} F_n(x_n) \geq F(x)$

(U) $\forall x \in X : \exists\, x_n \to x$ such that $\limsup_{n\to\infty} F_n(x_n) \leq F(x)$.

Proving Γ-convergence amounts to proving (L) and (U). (Here (L) stands for 'Lower limit' and (U) stands for 'Upper limit'.)

The theory of Γ-convergence then provides the following key result:

THEOREM 1 *Let $F_n \xrightarrow{\Gamma} F$ and let x_n minimize F_n. If x is a cluster point of $\{x_n\}$ then x minimizes F.*

(Under various regularity hypotheses, e.g., lower-semicontinuity of F, one can relax the condition that x_n be an exact minimizer of F_n.) This theorem says that all limit points of minimizers of F_n are minimizers of F. Thus, we can compute minimizers of F_n and, as n increases, if the solution converges then it converges to a minimizer of F. Similar remarks apply to local minimizers. This is what makes Γ-convergence practical: one can obtain approximations to minimizers of F without having to obtain explicit representations of them.

Γ-convergence is stable under various perturbations. In particular, if G is a continuous function on X and $F_n \xrightarrow{\Gamma} F$, then $G + F_n \xrightarrow{\Gamma} G + F$. If, moreover, $G \geq \epsilon > 0$, then $GF_n \xrightarrow{\Gamma} GF$.

In this paper the separable metric space X is the space of rectifiable curves of finite total absolute curvature, or turn, in $\mathbf{R}^n$, endowed with the metric defined by

$$d(\gamma_1, \gamma_2) = \inf_{\psi : [0,1]\to[0,1]} \sup_{t\in[0,1]} \left|\gamma_1(t) - \gamma_2(\psi(t))\right|$$

where ψ is a homeomorphism, i.e., a reparametrization, and we assume, without loss of generality, that γ_1 and γ_2 are defined on [0, 1]. (Formal definitions can be found in Section 2.)

We consider polygonal curves which, modulo a starting point, can be interpreted as a series of links or vectors $2\vec{\ell}_1, \ldots, 2\vec{\ell}_n$ in $\mathbf{R}^n$ (the reason for the factors of 2 will be apparent later). Let $A_{i,i+1}$ denote the exterior angle between $\vec{\ell}_i$ and $\vec{\ell}_{i+1}$ and let ℓ denote $\|\vec{\ell}\|$. We will define various functionals G on pairs of segments and consider functionals on polygons of the form

$$\sum_{i=1}^{n-1} G(\vec{\ell}_i, \vec{\ell}_{i+1}).$$

The critical issue is the choice of G. In general, for a given continuous domain functional which one wants to approximate, there will be more than one good choice for G. In our case of interest, approximating F_α, we will consider the three below.

$$G_\alpha^1(\vec{\ell}_1, \vec{\ell}_2) := \frac{A^\alpha}{(\ell_1 + \ell_2)^\alpha} \frac{\ell_1^{2\alpha-1} + \ell_2^{2\alpha-1}}{(\ell_1\ell_2)^{\alpha-1}}.$$

$$G_\alpha^2(\vec{\ell}_1, \vec{\ell}_2)$$
$$= \frac{A^\alpha |4\ell_1 - 2\ell_2|^{\alpha+1} \operatorname{sgn}(4\ell_1 - 2\ell_2) - |4\ell_2 - 2\ell_1|^{\alpha+1} \operatorname{sgn}(4\ell_2 - 2\ell_1)}{6(1+\alpha)(\ell_1 + \ell_2)^{2\alpha-1}(\ell_1 - \ell_2)}.$$

$$G_\alpha^3(\vec{\ell}_1, \vec{\ell}_2) := \frac{A^\alpha}{(\min(\ell_1, \ell_2))^{\alpha-1}}.$$

When α is an even integer, the case of most practical interest, G_α^2 simplifies; In particular, we have

$$G_2^2\left(\vec{\ell}_1, \vec{\ell}_2\right) = \frac{A^2}{(\ell_1 + \ell_2)} \frac{4(\ell_1^2 - \ell_1\ell_2 + \ell_2^2)}{(\ell_1 + \ell_2)^2}.$$

(Some readers may object to the explicit appearance of the angle A in these formulas since it is, in general, relatively expensive to compute. We remark that the substitution $A^2 \simeq (\ell_1\ell_2 - \langle\vec{\ell}_1, \vec{\ell}_2\rangle)/(\ell_1\ell_2)$ may be used without changing the main results.)

Our canonical problem is to find a curve γ which minimizes

$$F_\alpha(\gamma)$$

subject to boundary conditions specifying end points and end tangents. It is possible to consider more relaxed boundary conditions but extensions to such cases are straightforward. As a discrete approximation, we consider n segment polygons satisfying the same boundary conditions. We define the functionals

$$K_\alpha(\mathcal{P}) := \sum G(\vec{\ell}_i, \vec{\ell}_{i+1}),$$

with some modifications, and show that they Γ converge to F_α as the number of segments tends to ∞.

3 MATHEMATICAL PRELIMINARIES: THEORY OF CURVES

A *parameterized curve* is a continuous function $c : [a, b] \to \mathbf{R}^n$ that is non-constant on any open subinterval. A *curve* is an equivalence class of parameterized curves where a parameterized curve $c' : [d, e] \to \mathbf{R}^n$ is equivalent to c if there exists a homeomorphism $\psi : [c, e] \to [a, b]$ such that $\psi(d) = a$, $\psi(e) = b$, and $c' = c \circ \psi$.

Let $v_0, v_1, \ldots, v_n$ be the ordered vertices of a polygonal curve $\mathcal{P}$ and define $\vec{\ell}_i := (v_i - v_{i-1})/2$. Let $A_{i-1,i}$ denote the exterior angle formed by the ordered vertices v_{i-1}, v_i, v_{i+1}, i.e., the angle between $\vec{\ell}_i$ and $\vec{\ell}_{i+1}$, for $i = 1, \ldots, n-1$. The *length* of a polygonal curve $\mathcal{P}$ is defined as

$$L(\mathcal{P}) := \sum_{i=1}^{n} |v_i - v_{i-1}| = 2 \sum_{i=1}^{n} \ell_i,$$

where $|\cdot|$ denotes the Euclidean norm and we use ℓ_i to denote $|\vec{\ell}_i|$. The *total absolute curvature* or *turn* of $\mathcal{P}$ is defined as

$$K_1(\mathcal{P}) := \sum_{i=1}^{n} A_{i-1,i}.$$

A polygonal curve $\mathcal{P}$ is *inscribed* in a curve c if, under any parameterization $c(t)$, there exists $t_0 < t_1 < \cdots < t_n$ such that $v_i = c(t_i)$. We extend the definition of L and K_1 to arbitrary curves by

$$L(c) := \sup\{L(\mathcal{P}) : \mathcal{P} \text{ is inscribed in c}\},$$
$$K_1(c) := \sup\{K(\mathcal{P}) : \mathcal{P} \text{ is inscribed in c}\}.$$

These definitions are equivalent to the classical ones for smooth curves. This definition of total absolute curvature appeared first in [13].

If $L(c) < \infty$, then c is a *rectifiable* curve. Rectifiable curves admit an *arc-length parameterization*. This is the unique parameterization $c : [0, L(c)] \to \mathbf{R}^n$ such that $L(c|_{[0,t]}) = t$. This parameterization can be renormalized (linearly scaled) so that $L(c|_{[0,t]}) = tL(c)$, which is then called a *normalized* arc-length parameterization. Arc length parameterized rectifiable curves are absolutely continuous functions; therefore, they possess a derivative (of unit magnitude) almost everywhere. Let $\dot{c}(t)$ denote the unit tangent vector to c wherever it exists. Since we deal exclusively with rectifiable curves, we will usually not distinguish between a curve and its arc-length parameterization.

The Γ-convergence theory requires a separable metric space over which all functionals are defined. This means that we require a space of curves which includes polygons and smooth curves and over which a suitable metric is defined. A natural choice is the space of RFT curves: rectifiable curves of finite total absolute curvature endowed with a metric d defined as follows

$$d(\gamma_1, \gamma_2) = \inf_{\psi : [0,1] \to [0,1]} \sup_{t \in [0,1]} \left| \gamma_1(t) - \gamma_2(\psi(t)) \right|$$

where $\gamma_1(t)$, $t \in [0, 1]$ is any (canonical) parameterization of γ_1 and $\gamma_2(\psi(t))$ is any (canonical) parameterization of γ_2. This metric is a natural one: it measures how far apart two pencils would have to separate if the two curves were drawn simultaneously.

We now present some elementary theoretical results for RFT curves which establish their appropriateness for the problem at hand. The first several results appear without proof and can be found in [20]. Throughout this section, unless specified otherwise, $c(t)$ is an arc-length parameterized curve of finite total absolute curvature in $\mathbf{R}^n$.

THEOREM 2 *Let $c_i, i = 1, 2, \ldots$ be a sequence of RFT curves converging to an RFT curve c with respect to $d(\cdot, \cdot)$. Then*

$$L(c) \leq \liminf_{i \to \infty} L(c_i),$$

$$K_1(c) \leq \liminf_{i \to \infty} K_1(c_i).$$

THEOREM 3 *Let c_i be a sequence of arc-length parameterized curves satisfying $K(c_i) \leq M$, $L(c_i) \leq M$, and $\max_t |c_i(t)| \leq M$ for some $M < \infty$. Then either $\lim_{i\to\infty} L(c_i) = 0$ or there exists a subsequence $\{c_{i_k}\}$ and an arc-length parameterized curve c such that $\hat{c}_{i_k} \to \hat{c}$ uniformly, where $\hat{}$ denotes normalized arc-length parameterization. Furthermore, for any such subsequence, we have $L(c_{i_k}) \to L(c)$.*

The next lemma provides some regularity properties.

LEMMA 4 *If c is an RFT curve then $\dot{c}$ has right and left limits at t, $\dot{c}(t_+)$ and $\dot{c}(t_-)$, respectively, at all interior t and $\dot{c}$ has the appropriate one sided limit at its end points.*

Since c is absolutely continuous, it follows that $\dot{c}(t_-)$ and $\dot{c}(t_+)$ are also the right and left tangents of c at t, respectively. Thus, RFT curves have well defined endpoint tangents $\dot{c}(0+)$ and $\dot{c}(L(c)-)$ respectively. We will say a curve γ matches the first order endpoint conditions of c if the endpoints of the curves coincide and their tangents also coincide there.

The unit tangent curve $\dot{c}(t)$ of a curve $c(t)$ of finite total turn is a function of bounded variation. Thus, the second derivative of such curves can be represented as a measure. We will say a curve is weakly twice differentiable if the curvature measure is absolutely continuous with respect to arc-length. We will say that a curve c is a $W^{2,\alpha}$ curve if it is weakly twice differentiable and the Radon–Nikodyn derivative of the curvature measure, which we denote by $\ddot{c}$, has finite $L^\alpha[0, L(c)]$ norm. Thus, roughly speaking, $W^{2,2}$ curves are curves whose curvature has finite energy.

Given a curve RFT γ and a constant $1 \leq \alpha < \infty$ we define the functional F^α as follows. If γ is not weakly twice differentiable then $F^\alpha(\gamma) = \infty$ otherwise we set $F^\alpha(\gamma)$ to be the αth power of the $L^\alpha[0, L(c)]$ norm of $\ddot{\gamma}$, i.e.,

$$F^\alpha(\gamma) := \int_0^{L(\gamma)} |\ddot{\gamma}(s)|^\alpha ds$$

LEMMA 5 *If $\lim_{n\to\infty} d(\gamma_n, \gamma) = 0$ then*

$$F^\alpha(\gamma) \leq \liminf_{n\to\infty} F^\alpha(\gamma_n)$$

Proof Since $L(\gamma) \leq \liminf_{n\to\infty} L(\gamma_n)$ the lemma follows easily from standard lower-semicontinuity of L_α norms. □

4 APPROXIMATING ONE CURVE BY ANOTHER

The proof of Γ-convergence requires various approximations of curves of one regularity class by curves of another regularity class. In this section we develop all of the required approximation results. The approximations are of two types. The first type is directed toward proving the upper limit of Γ-convergence. Here we will approximate polygons by C^2 curves to obtain lower-semicontinuity results for them. The second type goes the opposite way: we approximate smooth curves by RFT curves.

4.1 Approximation of Polygons by Piecewise C^2 Curves

If γ is a smooth curve then $F_1(\gamma) = K(\gamma)$. Thus, K generalizes F_1. We desire to similarly extend the definition of F_α, $\alpha > 1$, to polygonal curves. The fundamental technical problem we must solve is how to assign a curvature functional to polygons that mimics and approximates F_α.

Our general strategy is the following. Given a polygonal curve P we will produce a piecewise C^2 curve γ that approximates P; it coincides with and is tangent to P at the midpoint of each segment and on the first and last half segments. Thus, each vertex and its adjoining segments can be considered separately. If $2\vec{\ell}_1$, $2\vec{\ell}_2$ are adjacent segments in P, then we approximate $[\vec{\ell}_1, \vec{\ell}_2]$ with a (piecewise) C^2 curve γ satisfying the first order endpoint conditions of $[\vec{\ell}_1, \vec{\ell}_2]$. We then define $G(\vec{\ell}_1, \vec{\ell}_2)$ in order to approximate $F_\alpha(\gamma)$. In all cases, when $\ell_1 = \ell_2$ the approximating curve will be a circular arc, or approximately so, at least in the case of small A, the angle between ℓ_1 and ℓ_2.

Let $x_1 = \vec{\ell}_1 + \vec{\ell}_2$ be the endpoints of $[\vec{\ell}_1, \vec{\ell}_2]$. Without loss of generality we may consider the plane of $\vec{\ell}_1$ and $\vec{\ell}_2$ to be the complex plane, and we may assume that the direction of the segment $\vec{\ell}_1$ is the direction of the positive real axis. (Our constructions will be invariant under rotation and translation.)

Construction 1

Let us first consider the equal length segment case: Let $\ell_1 = \ell_2 =: \ell$ and let $A \in (0, \pi)$ be the angle between $\vec{\ell}_1$ and $\vec{\ell}_2$. There is a unique circular arc $\Gamma_c(\vec{\ell}_1, \vec{\ell}_2)$ matching the first order endpoint conditions of $[\vec{\ell}_1, \vec{\ell}_2]$, see Fig. 2. The magnitude of the curvature of $\Gamma_c(\vec{\ell}_1, \vec{\ell}_2)$ is $\tan(A/2)/\ell$. Since the length of the arc is A times the length of the radius, we have

$$\begin{aligned} F_\alpha\Big(\Gamma_c(\vec{\ell}_1, \vec{\ell}_2)\Big) &= \frac{A(\tan A/2)^{\alpha-1}}{\ell^{\alpha-1}} \\ &= \frac{A^\alpha}{(2\ell)^{\alpha-1}}(1 + 0(A^2)). \end{aligned}$$

It follows trivially that $d(\Gamma_c(\vec{\ell}_1, \vec{\ell}_2), [\vec{\ell}_1, \vec{\ell}_2]) \le \ell$, but for $A \le \pi/2$ we have the stronger estimate

$$d\Big(\Gamma_c(\vec{\ell}_1, \vec{\ell}_2), [\vec{\ell}_1, \vec{\ell}_2]\Big) \le \ell \frac{1 - \cos A/2}{\sin A/2} \le \ell \frac{A}{4}.$$

We now consider the general case, i.e., we no longer assume $\ell_1 = \ell_2$. Let $\vec{v} = \vec{\ell}_1 + \vec{\ell}_2$. Consider the polygonal curve $[\lambda\vec{\ell}_1, (1-\lambda)\vec{v}, \lambda\vec{\ell}_2]$ where $\lambda \in (0,1)$. For a unique choice of λ we have $(1-\lambda)v = \lambda(\ell_1 + \ell_2)$. It is not difficult to solve for this λ: We have

$$(1-\lambda) = \frac{1 + \cos A}{1 + \cos A + \cos B + \cos C},$$

where B is the angle between $\vec{v}$ and $\vec{\ell}_1$ and $C = A - B$, see Fig. 3. Thus, we have $\lambda = \frac{1}{2} + 0(A^2)$.

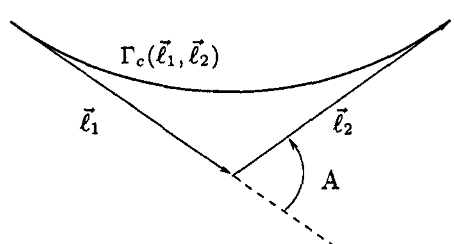

FIGURE 2 Circular arc fit of equal length line segments.

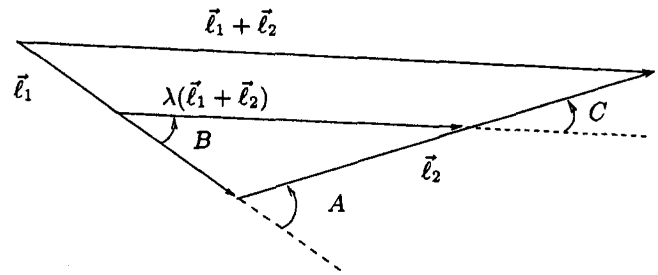

FIGURE 3 Framework for fitting piecewise circular C_2 curve.

By identifying the plane of the curve with the complex plane we can rewrite the polygonal curve $[\lambda\vec{\ell}_1, (1-\lambda)\vec{v}, \lambda\vec{\ell}_1]$ as

$$\left[\lambda\vec{\ell}_1, \lambda e^{iB}\vec{\ell}_1, \lambda e^{-iC}\vec{\ell}_2, \lambda\vec{\ell}_2\right],$$

i.e., as the concatenation of two equal length segments pairs. Thus, the piecewise C^2 curve

$$\Gamma^1(\vec{\ell}_1, \vec{\ell}_2) := \left[\Gamma_c\left(\lambda\vec{\ell}_1, \lambda e^{iB}\vec{\ell}_1\right), \Gamma_c\left(\lambda e^{-iC}\vec{\ell}_2, \lambda\vec{\ell}_2\right)\right]$$

matches the first order endpoint conditions of $[\vec{\ell}_1, \vec{\ell}_2]$. Moreover, we have

$$F_\alpha\left(\Gamma^1(\vec{\ell}_1, \vec{\ell}_2)\right) = \frac{B(\tan B/2)^{\alpha-1}}{(\lambda\ell_1)^{\alpha-1}} + \frac{C(\tan C/2)^{\alpha-1}}{(\lambda\ell_2)^{\alpha-1}}.$$

Since $\lambda = \frac{1}{2} + 0(A^2)$, $B = \ell_2/(\ell_1+\ell_2)A(1+0(A^2))$, and $A = B + C$ we obtain

$$\begin{aligned} F_\alpha\left(\Gamma^1(\vec{\ell}_1, \vec{\ell}_2)\right) &= \left[\frac{((\ell_2/(\ell_1+\ell_2))A)^\alpha}{\ell_1^{\alpha-1}} + \frac{((\ell_1/(\ell_1+\ell_2))A)^\alpha}{\ell_2^{\alpha-1}}\right](1+0(A^2)) \\ &= \frac{A^\alpha}{(\ell_1+\ell_2)^\alpha} \frac{\ell_1^{2\alpha-1} + \ell_2^{2\alpha-1}}{(\ell_1\ell_2)^{\alpha-1}}\left(1+0(A^2)\right). \end{aligned}$$

This motivates the definition

$$G_\alpha^1(\vec{\ell}_1, \vec{\ell}_2) := \frac{A^\alpha}{(\ell_1+\ell_2)^\alpha} \frac{\ell_1^{2\alpha-1} + \ell_2^{2\alpha-1}}{(\ell_1\ell_2)^{\alpha-1}}.$$

Construction 2

Note that the functional presented above is not, by any means, the only possible one. In the special case $\alpha = 2$ we have

$$G_2^1\left(\vec{\ell}_1, \vec{\ell}_2\right) = \frac{A^2}{(\ell_1 + \ell_2)} \frac{\ell_1^2 - \ell_1\ell_2 + \ell_2^2}{\ell_1\ell_2}.$$

The construction below leads to the following slightly different functional

$$G_2^2\left(\vec{\ell}_1, \vec{\ell}_2\right) = \frac{A^2}{(\ell_1 + \ell_2)} \frac{4(\ell_1^2 - \ell_1\ell_2 + \ell_2^2)}{(\ell_1 + \ell_2)^2}.$$

Let A be the exterior angle of $[\vec{\ell}_1, \vec{\ell}_2]$. If $A = 0$ then we set $\Gamma(\vec{\ell}_1, \vec{\ell}_2) = [\vec{\ell}_1, \vec{\ell}_2]$. Henceforth we assume $A > 0$. If A is large, $A \geq 3\pi/4$ say, then any reasonable choice of $\Gamma(\vec{\ell}_1, \vec{\ell}_2)$ will suffice. Henceforth, assume $0 < A \leq 3\pi/4$.

Let $\kappa(s) := a + bs$ with

$$a = A\frac{2(2\ell_2 - \ell_1)}{(\ell_1 + \ell_2)^2}, \quad b = A\frac{6(\ell_1 - \ell_2)}{(\ell_1 + \ell_2)^3}$$

and define

$$T(s) := e^{i\int_0^s \kappa(t)dt},$$
$$z(s) = \int_0^s T(t)dt.$$

The arc-length parameterized curve $z(s)$ satisfies $z(0) = 0$, $\dot{z}(0) = 1$, and $\dot{z}(\ell_1 + \ell_2) = e^{iA}$. Furthermore it is easily verified that $F_2(z) = \int_0^{\ell_1+\ell_2} \kappa^2(s)ds = G_2^2(\vec{\ell}_1, \vec{\ell}_2)$. More generally we may define

$$\begin{aligned} F_\alpha(z) &= \int_0^{\ell_1+\ell_2} |\kappa(s)|^\alpha \, ds \\ &= \frac{A^\alpha}{6(1+\alpha)(\ell_1+\ell_2)^{2\alpha-1}} \\ &\quad \times \frac{|4\ell_1 - 2\ell_2|^{\alpha+1}\operatorname{sgn}(4\ell_1 - 2\ell_2) - |4\ell_2 - 2\ell_1|^{\alpha+1}\operatorname{sgn}(4\ell_2 - 2\ell_1)}{(\ell_1 - \ell_2)}. \end{aligned}$$

Note that if α is an integer, then we can write

$$F_\alpha(z) = \frac{1}{6(1+\alpha)(\ell_1+\ell_2)^{2\alpha-1}} \frac{(4\ell_1 - 2\ell_2)^{\alpha+1} - (4\ell_2 - 2\ell_1)^{\alpha+1}}{(\ell_1 - \ell_2)}$$

and $(\ell_1 - \ell_2)$ is a factor of $(4\ell_1 - 2\ell_2)^{\alpha+1} - (4\ell_2 - 2\ell_1)^{\alpha+1}$.

Although z does not satisfy both end point conditions, we may perturb it to do so. Let λ_1 and λ_2 be defined by $z(\ell_1 + \ell_2) = \lambda_1 \vec{\ell}_1 + \lambda_2 \vec{\ell}_2$ and define the 2×2 matrix

$$M := \mathbf{m}\left(\vec{\ell}_1, \vec{\ell}_2\right) \begin{bmatrix} 1/\lambda_1 & \\ & 1/\lambda_2 \end{bmatrix} \mathbf{m}\left(\vec{\ell}_1, \vec{\ell}_2\right)^{-1}.$$

where $\mathbf{m}(\vec{\ell}_1, \vec{\ell}_2)$ denotes the 2×2 real matrix whose columns are the vectors $\vec{\ell}_1$ and $\vec{\ell}_2$ represented as real 2-dimensional vectors. It follows that the curve $\hat{z}(s)$ given by

$$\begin{bmatrix} \Re(\hat{z}(s)) \\ \Im(\hat{z}(s)) \end{bmatrix} := M \begin{bmatrix} \Re(z(s)) \\ \Im(z(s)) \end{bmatrix}$$

satisfies $\hat{z}(0) = 0$, $\dot{\hat{z}}(0) = 1/\lambda_1$, $\dot{\hat{z}}(\ell_1 + \ell_2) = (1/\lambda_2)e^{iA}$, and $\hat{z}(\ell_1 + \ell_2) = \vec{\ell}_1 + \vec{\ell}_2$. Thus $\hat{z}$ matches the endpoint conditions of $\vec{\ell}_1 \vec{\ell}_2$. Let us define

$$\Gamma^2(\vec{\ell}_1, \vec{\ell}_2) := \hat{z}.$$

We claim that

$$M = I + O(A).$$

Thus $\hat{z}$ is not very different from z when A is small, and in fact it follows that

$$F_\alpha(\hat{z}) = F_\alpha(z)(1 + O(A)).$$

We now prove the claim.

Consider $\varphi(s) := \int_0^s \kappa(t)dt = as + \frac{1}{2}bs^2$. It is easy to see that if $2\ell_2 \geq \ell_1$ then $0 \leq \varphi(s) \leq A$. If $2\ell_2 < \ell_1$ then we have $A\frac{2}{3}(\ell_1 - 2\ell_2)^2/(\ell_1^2 - \ell_2^2) \leq \varphi(s) \leq A$. In either case we have $|\varphi(s)| \leq A$. Thus, if A is small, then we have

$$e^{i\varphi(s)} = (1 + i\varphi(s))\big(1 + O(A^2)\big)$$

and hence

$$\begin{aligned} z(\ell_1 + \ell_2) &= [(\ell_1 + \ell_2) + i\ell_2 A]\big(1 + 0(A^2)\big) \\ &= (\vec{\ell}_1 + \vec{\ell}_2)\big(1 + O(A^2)\big). \end{aligned}$$

We now have $\lambda_1 = 1 + O(A^2)$ and $\lambda_2 = 1 + O(A^2)$ and since $\det \mathbf{m}(\vec{\ell}_1, \vec{\ell}_2) = \ell_1 \ell_2 \sin A$ we obtain

$$M = I + O(A).$$

Construction 3

Without loss of generality let $\ell_1 \leq \ell_2$. Now, consider the piecewise C_2 curve

$$\Gamma^3(\vec{\ell}_1, \vec{\ell}_2) := \left[\Gamma_c\left(\vec{\ell}_1, \frac{\ell_1}{\ell_2}\vec{\ell}_2\right), \left(1 - \frac{\ell_1}{\ell_2}\right)\vec{\ell}_2\right].$$

It is clear that $\Gamma^3(\vec{\ell}_1, \vec{\ell}_2)$ matches the first order endpoint conditions of $[\vec{\ell}_1 \vec{\ell}_2]$ and

$$\begin{aligned} F_\alpha\Big(\Gamma^3(\vec{\ell}_1, \vec{\ell}_2)\Big) &= F_\alpha\left(\Gamma_c\left(\vec{\ell}_1, \frac{\ell_1}{\ell_2}\vec{\ell}_2\right)\right) \\ &= \frac{A^\alpha}{(\ell_1)^{\alpha-1}}\big(1 + O(A^2)\big). \end{aligned}$$

This motivates the definition

$$G^3_\alpha(\vec{\ell}_1, \vec{\ell}_2) := \frac{A^\alpha}{(\min(\ell_1, \ell_2))^{\alpha-1}}.$$

Let the segments of a polygonal curve $\mathcal{P}$ be $2\vec{\ell}_1, 2\vec{\ell}_2, \ldots, 2\vec{\ell}_n$ with exterior angles $A_{12}, A_{23}, \ldots, A_{n-1,n}$. Now consider replacing each piece $[\vec{\ell}_i, \vec{\ell}_{i+1}]$ with the curve $\Gamma^j(\vec{\ell}_i, \vec{\ell}_{i+1})$, $j \in \{1, 2, 3\}$ and let γ^j be the resulting piecewise C^2 curve. It follows that

$$\begin{aligned}
F_\alpha(\gamma^j) &= \sum_{i=1}^{n-1} F_\alpha\Big(\Gamma(\vec{\ell}_i, \vec{\ell}_{i+1})\Big)(1 + O(A_{i,i+1})) \\
&= \sum_{i=1}^{n-1} G_\alpha^j(\vec{\ell}_i, \vec{\ell}_{i+1})\big(1 + O(A_{i,i+1})\big) \\
&= K_\alpha^j(\mathcal{P})(1 + O(A_{\max})),
\end{aligned}$$

where $A_{\max}$ is the maximum among the $A_{i,i+1}$.

4.2 Approximation of RFT Curves by Smooth Curves

Ultimately we are interested in approximating RFT curves with polygons. As an intermediate step we will first approximate them with smooth (C^2) curves. In Appendix 1 we prove the following.

LEMMA 6 *For any RFT curve γ and constant $\alpha \in [1, \infty)$ there exists a sequence $\{\gamma_n\}$ of smooth (e.g. C^2) curves such that γ_n matches the first order endpoint conditions of γ and such that*

$$\begin{aligned}
&d(\gamma_n, \gamma) \to 0 \\
&L(\gamma_n) \to L(\gamma) \\
&F_\alpha(\gamma_n) \to F_\alpha(\gamma).
\end{aligned}$$

In Appendix 2 we prove the following.

LEMMA 7 *For any C_2 curve γ there exists a sequence of equal length segment polygons $\mathcal{P}_n$, with segment length $\ell_n \to 0$, such that each polygon $\mathcal{P}_n$ matches the first order endpoint conditions of γ and*

$$\lim_{n\to\infty} \sum_{i=0}^{m_n-1} \frac{A_{i,i+1}^\alpha}{\ell_n^{\alpha-1}} = F_\alpha(\gamma),$$

where m_n is the number of segments in $\mathcal{P}_n$. Furthermore, we have $L(P_n) \to L(\gamma)$ and $d(P_n, \gamma) \to 0$.

5 PROOF OF Γ-CONVERGENCE

We are now in a position to prove the essential ingredients of Γ-convergence, i.e., to prove (L) and (U). First we prove (L). This requires constructing good polygonal approximations to curves with respect to the given metric and the functional to be approximated. It turns out that we can obtain good approximations by focussing on polygons consisting of equal length segments.

LEMMA 8 *Let $G(\vec{\ell}_1, \vec{\ell}_2)$ satisfy*

$$\lim_{\ell \to 0} \left\{ G(\vec{\ell}_1, \vec{\ell}_2) : \ell_1 = \ell_2 = \ell, |A| > \delta > 0 \right\} = \infty$$

$$\lim_{\substack{A \to 0 \\ \ell \to 0}} \left\{ G(\vec{\ell}_1, \vec{\ell}_2) \frac{\ell^{\alpha-1}}{|A|^\alpha} : \ell_1 = \ell_2 = \ell \right\} = 1.$$

Then for any $W^{2,\alpha}$ curve γ there exists a sequence of polygonal curves P_n, each satisfying the same first order endpoint conditions as γ, such that

(i) $d(P_n, \gamma) \to 0$
(ii) $L(P_n) \to L(\gamma)$
(iii) $K_\alpha(P_n) \to F_\alpha(\gamma)$.

Proof Let γ_j be a sequence of C^2 curves satisfying (i), (ii), and (iii) in place of P_n. For each j there exist a sequence P_{jn} such that (i), (ii), and (iii) are satisfied with P_{jn} replacing P_n and γ_j replacing γ. By a standard diagonalization argument we can extract a subsequence satisfying (i), (ii), (iii). □

We are now ready to prove part (U) under the assumption that the segment lengths tend to zero. We will address this issue further in the next section.

LEMMA 9 *Let $\mathcal{P}_n$ be a sequence of polygons satisfying $d(\mathcal{P}_n, \gamma) \to 0$ and such that the maximum segment length in $\mathcal{P}_n$ tends to 0. Then*

$$F_\alpha(\gamma) \le \liminf_{n \to \infty} K_\alpha^j(P_n)$$

for $j \in \{1, 2, 3\}$.

Proof For each j we find a piecewise C^2 curve γ_n according to the constructions above approximating P_n. Without loss of generality we assume $\lim K_\alpha^j(P_n)$ exists and is finite. It follows that the maximum angle between adjacent segments in P_n tends to 0. Hence $d(\gamma_n, P_n) \to 0$ and $|K_\alpha(P_n) - F_\alpha(\gamma_n)| \to 0$. Since $\gamma_n \in W^{2,\alpha}$ and $d(\gamma_n, \gamma) \to 0$ we have

$$F_\alpha(\gamma) \leq \liminf_{n\to\infty} F_\alpha(\gamma_n) = \lim_{n\to\infty} K_\alpha(P_n). \qquad \square$$

6 APPLICATIONS

In this section we apply our results. Note that most of our convergence results require that the maximum length of a segment in the approximation tend to zero. This natural condition is actually necessary. Consider the (polygonal curve) $\gamma = [\vec{\ell}_1, \vec{\ell}_2]$. We have $F_\alpha(\gamma) = +\infty$ but without requiring segment lengths to tend to zero we can duplicate γ with a multi-segment polygon while keeping K_α bounded. Thus, the maximum length condition should be enforced in order to obtain Γ-convergence results. In the constructions of [14] this was accomplished by using only equal segment polygons. Another way to enforce this is to append another term to the functional. There are many possibilities; a simple example is $\text{Reg}(\mathcal{P}) := n\sum_{i=1}^{n}(\ell_i - \ell_{i+1})^2$. Note that this functional is zero on equal segment length polygons but does not force all lengths to be asymptotically equal since it tends to zero as long as the maximum segment length is $O(1/n)$. As long as the length of $\mathcal{P}$ is bounded below by a positive constant, then $\text{Reg}(\mathcal{P})$ will be grow to ∞ unless the maximum segment length tends to zero.

Consider the following problem: Given first order endpoint conditions, find the curve γ which minimizes

$$\Theta(L(\gamma), F_\alpha(\gamma)),$$

where $\Theta(x, y)$ is any continuous function satisfying $\lim_{x\to\infty} \Theta(x, y) = \infty$.

Now consider the following discrete problem. Find the n segment polygon which minimizes

$$\Theta(L(\mathcal{P}), K_{\alpha,n}(\mathcal{P})) + \text{Reg}(\mathcal{P})$$

where

$$K_{\alpha,n}(\mathcal{P}) := \sum_{i=1}^{n-1} G_\alpha^j(\vec{\ell}_i, \vec{\ell}_{i+1})$$

for $j = 1, j = 2$ or $j = 3$.

If we extend both functionals to RFT curves by setting them to ∞ on curves for which they are not intrinsically well defined, then we have

$$\Theta(L(), K_{\alpha,n}()) + \mathrm{Reg}() \xrightarrow{\Gamma} \Theta(L(), F_\alpha()).$$

Note that this holds true if we restrict to equal segment length polygons and, in this case, we can remove the redundant term $\mathrm{Reg}(\mathcal{P})$.

This example covers all of the original functionals (Section 1) provided the endpoint conditions guarantee that any finite solution must have positive length.

As a final example we consider the problem of Dubins [18]. Given first order endpoint conditions, Dubins's problem is to find the curve γ (in the plane) of shortest length satisfying the endpoint conditions that has curvature no greater than K. We can approximate this cost functional with a smooth functional as follows. Consider the functional

$$L(\gamma) + F_\alpha(\gamma)/K^\alpha = L(\gamma) + ((F_\alpha(\gamma))^{1/\alpha}/K)^\alpha$$

on curves γ satisfying the endpoint conditions. If we let $\alpha \to \infty$ then, for twice weakly differentiable γ, this functional converges to $L(\gamma)$ or $+\infty$ for L_∞ norm of $\ddot{\gamma}$ less than or more than K, respectively. The case $\|\gamma\|_\infty = K$ is, unfortunately, a little problematic. However, a slight variation of the functional such as below,

$$H_\alpha(\gamma) := L(\gamma) + F_\alpha(\gamma)/(K + 1/\sqrt{\alpha})^\alpha = L(\gamma) + ((F_\alpha(\gamma))^{1/\alpha}/(K + 1/\sqrt{\alpha}))^\alpha$$

guarantees convergence to $L(\gamma)$ if $\|\gamma\|_\infty = K$.

Letting α be integer, the functionals above Γ-converge to Dubins's functional. It follows that if we have sequences $H_{\alpha,n}$ which Γ-converge in n to H_α, then by a suitable diagonalization there exists a sequence $H_{\alpha_n,n}$ which Γ-converges to Dubins's functional. The dependence of α_n on n depends on the relative rates on convergence.

Thus, we may provably approximate Dubins's problem on polygonal curves.

6.1 A Computational Lifting Scheme

If we restrict to polygons of equal length segments then the polygon is easily approximated using circular arcs. Thus, under this restriction Γ-convergence is easily proved. If we wish to find a "good" functional $G(\vec{\ell}_1, \vec{\ell}_2)$ for some particular functional and α then we may use the equal length segment approximation to compute an approximation for the more general case.

For example, for some fixed $\vec{\ell}_1$ we can compute for various $\vec{\ell}_2$ a polygon locally minimizing $K_\alpha + \beta L$, where βL is necessary for regularization of the functional. Having sampled the functional K_α for various values of $\vec{\ell}_2$ we may attempt to find an analytic expression to approximate K_α especially in the regime of small A. We can use this lifting approach to build good approximations of various functionals.

7 CONCLUSIONS

We have shown that it is feasible to find good piecewise linear approximations to various types of planar elastica curves by first discretizing the problem itself. This is a viable approach to the practical design of splines, as already exemplified in [11] and [14]. The theory of Γ-convergence shows that we can get better and better approximations to the 'continuous' solutions of the original elastica problems by increasing the number of linear segments allowed in the polygonal approximations.

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APPENDIX 1: APPROXIMATION OF RFT CURVES BY SMOOTH CURVES

This section is devoted to proving Lemma 6 which we restate here for the reader's convenience.

LEMMA 6 *For any RFT curve γ and constant $\alpha \in [1, \infty)$ there exists a sequence $\{\gamma_n\}$ of smooth (e.g. C^2) curves such that γ_n matches the first order endpoint conditions of γ and such that*

$$\begin{aligned} d(\gamma_n, \gamma) &\to 0 \\ L(\gamma_n) &\to L(\gamma) \\ F_\alpha(\gamma_n) &\to F_\alpha(\gamma). \end{aligned}$$

Proof If γ is not a $W^{2,\alpha}$ curve then the result is trivial so we assume $\gamma \in W^{2,\alpha}$. Let f_n be a sequence of smooth functions converging to $\ddot{\gamma}$ in $L_\alpha\, [0, L(\gamma)]$. Define

$$F_n(s) := \int_0^s f_n(t)dt + \dot{\gamma}(0)$$
$$\bar{F}_n(s) := \int_0^s F_n(t)dt + \gamma(0).$$

Applying the Hölder inequality we obtain

$$|\dot{\gamma}(s) - F_n(s)| \leq s^{\alpha'} \|\ddot{\gamma} - f_n\|_\alpha$$
$$|\gamma(s) - \bar{F}_n(s)| \leq \left(\frac{s^3}{3}\right)^{\alpha'} \|\ddot{\gamma} - f_n\|_\alpha$$

where α' is defined by $1/\alpha' + 1/\alpha = 1$. For n large enough we have $F_n(s) \neq 0$ hence $\bar{F}_n(s)$ is a parameterized curve.

It is clear that $\bar{F}_n$ matches the first order end condition of γ at $s = 0$. To complete the proof we modify F_n so that it also matches the first order end condition of γ at $s = L(= L(\gamma))$. To this end we define

$$g_n(s) := f_n(s) + \frac{\dot{\gamma}(L) - F_n(L)}{L} + \frac{12}{L^3}\left(\frac{L}{2} - s\right)$$
$$\times \left(\gamma(L) - \bar{F}_n(L) - \frac{L}{2}(\dot{\gamma}(L) - F_n(L))\right)$$

and set

$$G_n(s) := \int_0^s g_n(t)dt + \dot{\gamma}(0),$$
$$\bar{G}_n(s) := \int_0^s G_n(t)dt + \gamma(0).$$

A straightforward calculation verifies that $\bar{G}_n(L) = \gamma(L)$ and $G_n(L) = \dot{\gamma}(L)$. Furthermore, we have

$$|\dot{\gamma}(s) - G_n(s)| \leq \text{const } \|\ddot{\gamma} - f_n\|_\alpha,$$
$$|\gamma(s) - \bar{G}_n(s)| \leq \text{const } \|\ddot{\gamma} - f_n\|_\alpha,$$

so that $\bar{G}_n(s)$ is a parameterized curve for n large enough. Assuming this is the case we set $\gamma_n = \bar{G}_n$. Since

$$L(\gamma_n) = \int_0^L |G_n(s)| ds$$

we obtain $L(\gamma_n) \to L(\gamma)$. Let t denote arc-length along γ_n; define $t(s) = \int_0^s |G_n(u)| du$, and let $s(t)$ denote the inverse of $t(s)$. Then

$$\begin{aligned}\ddot{\gamma}_n(t) &= \frac{d^2}{dt^2}\gamma_n(t) = \frac{d}{dt}\left(G_n(s(t))\frac{ds(t)}{dt}\right) = \frac{d}{dt}\frac{G_n(s(t))}{|G_n(s(t))|} \\ &= \left(\frac{g_n(s(t))}{|G_n(s(t))|} - \frac{G_n(s(t))}{|G_n(s(t))|^3}\langle G_n(s(t)), g_n(s(t))\rangle\right)\frac{ds(t)}{dt}.\end{aligned}$$

We have $G_n(s) \to \dot{\gamma}(s)$ uniformly and $g_n(s) \to \ddot{\gamma}(s)$ in L_α, so $\langle G_n(s), g_n(s)\rangle \to 0$ in L_α and $dt(s)/ds$, $ds(t)/dt$, and $|G_n(s)|$ all converge to 1 uniformly. We conclude $\ddot{\gamma}_n(t) - g_n(s(t)) \to 0$ in L_α and, moreover,

$$F_\alpha(\gamma_n) - \int_0^L |g_n(s)|^\alpha ds \to 0.$$

Hence $F_\alpha(\gamma_n) \to F_\alpha(\gamma)$. □

APPENDIX 2: APPROXIMATION OF C^2 CURVES BY POLYGONS

We now present the proof of Lemma 7, restated here for the reader's convenience.

LEMMA 7 *For any C_2 curve γ there exists a sequence of equal length segment polygons $\mathcal{P}_n$, with segment length $\ell_n \to 0$, such that each $\mathcal{P}_n$ matches the first order endpoint conditions of γ and*

$$\lim_{n\to\infty} \sum_{i=0}^{m_n-1} \frac{A^\alpha_{i,i+1}}{\ell_n^{\alpha-1}} = F_\alpha(\gamma),$$

where m_n is the number of segments in $\mathcal{P}_n$. Furthermore, we have $L(P_n) \to L(\gamma)$ and $d(P_n, \gamma) \to 0$.

Proof Given any endpoint conditions it is easy to construct polygons with $n \geq 5$ equal length segment satisfying the endpoint conditions. We will now give a construction approximating γ, which is valid for all n sufficiently large.

Let γ be C^2 curve with maximum curvature magnitude M, i.e., $|\ddot{\gamma}| \leq M$. Given a small positive constant δ, we will construct a polygon $\mathcal{P}$ consisting of equal length segments of length $\ell \leq \delta$ such that $\mathcal{P}$ matches the first order endpoint conditions of γ and all except the second and penultimate vertices of $\mathcal{P}$ lie on γ. Our purpose is to show that $F_\alpha(\gamma)$ can be well approximated by $K_\alpha(P)$ when δ is small. Furthermore, we will have $\ell/\delta \to 1$ as $\delta \to 0$, and the number of segments required will increase monotonically in steps of size one as δ is decreased.

Without loss of generality we assume $\delta \leq \min(\frac{1}{4}\ \mathrm{diam}(\gamma), \frac{1}{4}L(\gamma), 1/(4M+1))$. Setting $\ell = \delta$ define $v_0 = \gamma(0)$, $v_1 = \ell\dot{\gamma}(0) + v_0$, $w_0 = \gamma(1)$, and $w_1 = w_0 - \ell\dot{\gamma}(1)$. Define $t_2 = \min\{t > 0 : \|\gamma(t) - v_1\| \geq \ell\}$. Similarly, assuming it exists, define $t_i = \min\{t > t_{i-1} : \|\gamma(t) - v_{i-1}\| \geq \ell\}$ and define $t_2' = \max\{t' < L(\gamma)\colon \|\gamma(t') - w_1\| \geq \ell\}$. It can be easily seen that there is some smallest m such that $t_m \geq t_2'$. If $t_m = t_2'$ we define

$$\mathcal{P} = [v_0, v_1, \ldots, v_m(= w_2), v_{m+1}(= w_1), v_{m+2}(= w_0)],$$

and we are done. If $t_m > t_2'$ then we continuously decrease ℓ ($\ell < \delta$) until we have $t_m = t_2'$. It is easy to see that both t_m and t_2' are continuous monotonic functions of ℓ. Moreover, as δ decreases the number of segments required increases in discrete jumps of size one.

We will now examine how well $F_\alpha(\gamma)$ and $L(\gamma)$ can be estimated from $\mathcal{P}$. Let s denote arc-length along $\mathcal{P}$ and let t denote arc-length along γ. Let $s_0 < s_1 < \cdots < s_{m+2}$ be defined by $\mathcal{P}(s_i) = v_i$ (where $v_{m+1} = w_1$, $v_{m+2} = w_0$). We define a piecewise linear homeomorphism $\psi\colon [0, L(\gamma)] \to [0, L(\mathcal{P})]$ as follows: For $s = s_i$, $i = 0, 2, 3, \ldots, m, m+2$ we set

$$t_i = \psi(s_i)$$

and inbetween we linearly interpolate. We claim that for each $i = 2, \ldots, m-2$ we have

$$\ell \leq t_{i+1} - t_i \leq \ell\left(1 + \frac{M^2\ell^2}{6}\right). \tag{6}$$

The left inequality is trivial. To prove the right inequality we first prove the following

$$|\gamma(t) - \gamma(s)| \geq |t - s| - \frac{M^2}{12}|t - s|^3.$$

Without loss of generality we assume $t \geq s$. The angle θ formed between $\dot{\gamma}(t)$ and $\dot{\gamma}(s)$ is bounded above by $M(t - s)$. Now, $\langle \dot{\gamma}(t), \dot{\gamma}(s) \rangle = \cos\theta \geq 1 - \theta^2/2$, so we obtain

$$\begin{aligned}
|\gamma(t) - \gamma(s)| &= \left| \int_s^t \dot{\gamma}(u)du \right| \\
&\geq \int_s^t \left\langle \dot{\gamma}(u), \dot{\gamma}\left(\frac{t-s}{2}\right) \right\rangle du \\
&\geq \int_s^t \left(1 - \frac{M^2(u - (t-s)/2)^2}{2} \right) du \\
&= (t - s) - \frac{M^2}{12}(t - s)^3.
\end{aligned}$$

By assumption we have $M\ell \leq \frac{1}{4}$. Therefore, if $t - s = \ell\big(1 + (M^2\ell^2/6)\big)$, then

$$\begin{aligned}
|\gamma(t) - \gamma(s)| &\geq \ell\left(1 + \frac{M^2\ell^2}{6}\right)\left(1 - \frac{1M^2\ell^2}{12}\left(1 + \frac{M^2\ell^2}{6}\right)^2\right) \\
&\geq \ell\left(1 + \frac{M^2\ell^2}{6}\right)\left(1 - \frac{M^2\ell^2}{11}\right) \\
&\geq \ell.
\end{aligned}$$

From this the claim (6) follows.

For any $a \in [0, \ell]$ we have

$$F_\alpha(\gamma) \geq \int_{\psi(s_2+a)}^{\psi(s_{m-1}+a)} |\ddot{\gamma}|^\alpha ds$$

so we have

$$\begin{aligned}
F_\alpha(\gamma) &\geq \frac{1}{\ell}\int_0^\ell \left(\int_{\psi(s_2+a)}^{\psi(s_{m-1}+a)} |\ddot{\gamma}(t)|^\alpha dt \right) da \\
&= \sum_{i=3}^{m-1} \frac{1}{\ell} \int_0^\ell \left(\int_{\psi(s_{i-1}+a)}^{\psi(s_i+a)} |\ddot{\gamma}(t)|^\alpha dt \right) da.
\end{aligned}$$

By Jensen's inequality we have

$$\begin{aligned}\int_{\psi(s_{i-1}+a)}^{\psi(s_i+a)} |\ddot{\gamma}(t)|^\alpha dt &\geq (\psi(s_i+a)-\psi(s_{i-1}+a))^{1-\alpha}\left|\int_{\psi(s_{i-1}+a)}^{\psi(s_i+a)} \ddot{\gamma}(t)dt\right|^\alpha \\ &= (\psi(s_i+a)-\psi(s_{i-1}+a))^{1-\alpha} \\ &\quad \times \left|\dot{\gamma}(\psi(s_i+a))-\dot{\gamma}(\psi(s_{i-1}+a))\right|^\alpha \\ &\geq (\max(t_{i+1}-t_i, t_i-t_{i-1})))^{1-\alpha} \\ &\quad \times \left|\dot{\gamma}(\psi(s_i+a))-\dot{\gamma}(\psi(s_{i-1}+a))\right|^\alpha .\end{aligned}$$

And, again by Jensen's inequality, we have

$$\begin{aligned}&\int_0^\ell |\dot{\gamma}(\psi(s_i+a))-\dot{\gamma}(\psi(s_{i-1}+a))|^\alpha da \\ &\quad \geq \ell^{1-\alpha}\left|\int_0^\ell (\dot{\gamma}(\psi(s_i+a))-\dot{\gamma}(\psi(s_{i-1}+a)))da\right|^\alpha .\end{aligned}$$

Now, note that

$$\int_0^\ell \dot{\gamma}(\psi(s_i+a))\frac{d\psi}{da}da = \gamma(t_{i+1})-\gamma(t_i),$$

and

$$\frac{\partial\psi}{\partial a}(s_i+a) = \frac{t_{i+1}-t_i}{s_{i+1}-s_i},$$

so we obtain

$$\frac{1}{\ell}\int_0^\ell \dot{\gamma}(\psi(s_i+a))da = \frac{\gamma(t_{i+1})-\gamma(t_i)}{t_{i+1}-t_i} = \frac{\vec{\ell}_{i+1}}{t_{i+1}-t_i}.$$

Putting the above together we have

$$\begin{aligned}&\frac{1}{\ell}\int\int_{\psi(s_{i-1}+a)}^{\psi(s_i+a)} |\ddot{\gamma}(t)|^\alpha dt\, da \\ &\quad \geq (\max(t_{i+1}-t_i, t_i-t_{i-1}))^{1-\alpha}\left|\frac{\vec{\ell}_{i+1}}{t_{i+1}-t_i}-\frac{\vec{\ell}_i}{t_i-t_{i-1}}\right|^\alpha .\end{aligned}$$

Without loss of generality we assume $t_{i+1} - t_i \geq t_i - t_{i-1}$. We now have

$$\frac{1}{\ell}\int\int_{\psi(s_{i-1}+a)}^{\psi(s_i+a)} |\ddot{\gamma}(t)|^\alpha \, dt\, da \geq \frac{\ell^\alpha}{(t_{i+1}-t_i)^{2\alpha-1}}\left|\frac{\vec{\ell}_{i+1}}{\ell} - \frac{\vec{\ell}_i}{\ell} + \frac{\vec{\ell}_i}{\ell}\left(1 - \frac{t_{i+1}-t_i}{t_i - t_{i-1}}\right)\right|^\alpha.$$

Applying equation (6) we now obtain

$$\begin{aligned}\frac{1}{\ell}\int\int_{\psi(s_{i-1}+a)}^{\psi(s_i+a)} |\ddot{\gamma}(t)|^\alpha \, dt &\geq \frac{\ell^{1-\alpha}}{(1+M^2\ell^2)^{2\alpha-1}} 2^{\alpha/2}(1-\cos A_{i,i+1})^{\alpha/2} \\ &\quad \times (1 - M^2\ell^2)^{\alpha/2} \\ &\geq \frac{A_{i,i+1}^\alpha}{\ell^{\alpha-1}} \frac{(1-M^2\ell^2)^{\alpha/2}}{(1+M^2\ell^2)^{2\alpha-1}}.\end{aligned}$$

We have now shown

$$F_\alpha(\gamma) \geq \sum_{i=2}^{m-2} \frac{A_{i,i+1}^\alpha}{\ell^{\alpha-1}}\left(1 + 0(\ell^2)\right).$$

Note that had we inscribed P the limits of the sum would be 0 and m. However, it is easy to show that there is a constant c depending only on M such that if δ is sufficiently small then $A_{i,i+1}^\alpha/\ell^{\alpha-1} \leq c\ell$. We have therefore proven Lemma 7. □