Differential Methods in Inverse Scattering
Author(s): Alfred M. Bruckstein, Bernard C. Levy and Thomas Kailath
Source: *SIAM Journal on Applied Mathematics*, Vol. 45, No. 2 (Apr., 1985), pp. 312-335
Published by: Society for Industrial and Applied Mathematics
Stable URL: http://www.jstor.org/stable/2101819
Accessed: 15-12-2017 16:58 UTC

REFERENCES
Linked references are available on JSTOR for this article:
http://www.jstor.org/stable/2101819?seq=1&cid=pdf-reference#references_tab_contents
You may need to log in to JSTOR to access the linked references.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at http://about.jstor.org/terms

Society for Industrial and Applied Mathematics is collaborating with JSTOR to digitize, preserve and extend access to *SIAM Journal on Applied Mathematics*

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

SIAM J. APPL. MATH.
Vol. 45, No. 2, April 1985

© 1985 Society for Industrial and Applied Mathematics
009

DIFFERENTIAL METHODS IN INVERSE SCATTERING*

ALFRED M. BRUCKSTEIN†§, BERNARD C. LEVY‡ AND THOMAS KAILATH†

Abstract. This paper discusses a new set of differential methods for solving the inverse scattering problem associated to the propagation of waves in an inhomogeneous medium. By writing the medium equations in the form of a two-component system describing the interaction of rightward and leftward propagating waves, the causality of the propagation phenomena is exploited in order to identify the medium layer by layer. The recursive procedure that we obtain constitutes a continuous version of an algorithm first derived by Schur in order to test for the boundedness of functions analytic inside the unit circle. It recovers the local reflectivity function of the medium. Using similar ideas, some other differential methods can also be derived to reconstruct alternative parametrizations of the layered medium in terms of the local impedance or of the potential function.

The differential inverse scattering methods turn out to be very efficient since, in some sense, they let the medium perform the inversion by itself and thus fully exploit its structure. They provide an alternative to classical methods based on integral equations, which, in order to exploit the structure of the problem, must ultimately resort to differential equations of the same type.

1. Introduction. The inverse problem for the one-dimensional Schrödinger equation and for two-component scattering systems has received a large amount of attention over the years. This interest is motivated by the numerous applications of such problems in fields as varied as quantum physics, geophysics, transmission-line analysis, filter design, voice synthesis, see e.g. [1]–[13].

The first complete solution of the inverse scattering problem was obtained by Gelfand and Levitan [14], in the context of reconstructing a second order differential operator from its spectral function. Subsequently, several alternative solutions were proposed by Marchenko [15], Krein [16], Kay and Moses [17] and Faddeev [1], [18]. Other inversion procedures were derived by Gopinath and Sondhi [6], [7] and by Zakharov and Shabat [19], [20] for systems described respectively by transmission-line type equations and by two-component scattering models.

Since all the inverse scattering procedures mentioned above were formulated in terms of integral equations, it was widely accepted in the scientific community that inverse problems require the solution of such equations. However, independently of the work of mathematicians and physicists, geophysicists such as Goupillaud and Robinson (see e.g. [9], [10]) developed approaches that more directly exploited the physical properties of layered media in which waves propagate. Their solutions, sometimes referred to as dynamic deconvolution methods [9], reconstruct the medium in a recursive manner, layer by layer. However this work was formulated in terms of a discretized layered earth model and was therefore not recognized as providing a solution to the general inverse scattering problem. In fact, when dealing with continuously varying media, even geophysicists were relying on integral equations based approaches (see e.g. [4], [13], [21]).

However, more recently, Deift and Trubowitz [22] proposed a potential reconstruction method based on a trace formula, which calls for the propagation of an ordinary

* Received by the editors July 14, 1983, and in revised form July 30, 1984.

† Information Systems Laboratory, Stanford University, Stanford, California 94305. The work of these authors was supported in part by the U.S. Army Research Office, under contract DAAG29-79-C-0215 and by the Joint Services Program, under contract DAAG29-81-K-0057.

‡ Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139. The work of this author was supported by the Air Force Office of Scientific Research, under grant AFOSR-82-0135.

§ Present address, Faculty of Electrical Engineering, Technion, Haifa, Israel.

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

differential equation and thus departs from the classical inverse scattering framework. Also several other researchers, for example Symes [23], Santosa and Schwetlick [24], Sondhi and Resnick [25] and Bube and Burridge [26] have independently proposed and analyzed differential-equations-based methods whose relations to each other have not been clarified till now.

The objective of this paper is to give a comprehensive and unified account of differential inverse scattering methods, though we are not yet able to include the Deift–Trubowitz method in our framework. Our approach is based on first deriving an infinitesimal layer peeling procedure, which can be viewed as a continuous version of the dynamic deconvolution algorithm. The inversion methods analyzed by Bube and Burridge [26] and by Symes [23], as well as the impedance reconstruction method proposed by Santosa and Schwetlick [24] and by Sondhi and Resnick [25] for solving acoustical inverse problems, can all be interpreted from this point of view. The dynamic deconvolution algorithm is in fact a continuous form of a method used by Schur to test for the boundedness of functions analytic outside the unit disc [27], [28]. The identification of the Schur algorithm as a recursive layer extraction method was first made by Dewilde and his coworkers [29], [30].

The relation between the differential inverse scattering methods that we present and the classical integral-equations-based approaches is then discussed. The layer peeling methods are based on properly exploiting the differential structure of the medium and the causality of propagation through it. On the other hand, the integral equations of the classical inversion approaches will be shown to follow as a simple consequence of the symmetry, losslessness and causality properties of the scattering medium. It turns out that the variety of integral equations (e.g. Gelfand–Levitan, Marchenko, Krein) arises by considering special choices of input-output (excitation-response) pairs for the medium. In § 5, on the other hand, we shall first derive an apparently novel integral equation (5.16) corresponding to a general input-output pair. While all the integral equations can be solved by traditional numerical methods, the special properties of the scattering medium that gives rise to integral equations endow their kernels with special structure (e.g. Toeplitz + Hankel, Hankel or Toeplitz). This structure can be exploited to derive "fast" algorithms of the so-called Krein–Levinson type [31], [32], as was done in [33]. The Krein–Levinson equations have the same differential structure as the Schur type algorithms of § 3, but differ in the way certain "boundary data" are computed. These boundary conditions are computed using the original scattering data (the excitation-response pair) via inner products. These inner products, or integrations are avoided in the direct, Schur procedure, which is an advantage for parallel computation. The Schur form of solution can also be obtained directly from the integral equations, as shown by Gohberg and Koltracht [34], who also demonstrate the forward stability of both types of fast algorithms.

Although the main objective of this paper is to integrate and reinterpret a large number of earlier results of inverse scattering theory in a unified framework, a number of new results are also presented. These include an inversion method based on a Riccati differential equation with associated limit conditions in § 3.3, derivation of a new integral equation (5.16) as mentioned above and the extension of layer-peeling ideas to the study of general, nonlossless media in § 6.

The paper is organized as follows. Several physical models of a scattering medium are presented in § 1. These provide various equivalent parametrizations of the medium and give rise to different formulations of inverse scattering problems. The continuous layer-peeling algorithm and the associated Schur recursions for reconstructing the reflectivity function parametrization of the medium are derived in § 3. They are then

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

used in § 4 to obtain other differential methods that reconstruct the local impedance or the Schrodinger potential. Section 5 relates these differential methods to the integral equations approaches and describes the Krein–Levinson type differential solution of the inverse problem. In § 6 the results of the earlier sections are extended to some cases of lossy scattering media and § 7 concludes with observations on possible extensions of these results.

2. Physical models of scattering media. The inverse scattering methods we discuss in this paper concern several classes of physical models that correspond to equivalent descriptions of a lossless scattering medium. They arise in the study of transmission-lines and of vibrating strings, in the analysis of layered acoustic media and of the vocal tract and in the description of particle scattering in quantum physics [1]–[9], [35], [36].

The first model that we consider is described by the symmetrized *telegrapher's equations*

$$\frac{\partial}{\partial x}\begin{bmatrix} v(x,t) \\ i(x,t) \end{bmatrix} = \begin{bmatrix} 0 & -Z(x)\dfrac{\partial}{\partial t} \\ -Z(x)^{-1}\dfrac{\partial}{\partial t} & 0 \end{bmatrix}\begin{bmatrix} v(x,t) \\ i(x,t) \end{bmatrix} \tag{2.1}$$

which may be viewed as obtained from the usual transmission-line equations by assuming that the inductance per unit length equals the inverse of the capacitance. $Z(x)$ in the above equation corresponds to the *local impedance* for a transmission-line or to the *area function* of the vocal tract model [6], [24], [25]. Since in equation (2.1) the "voltage" and "current" variables are expressed in different units, we also consider the normalized quantities

$$V(x,t) = v(x,t)Z(x)^{-1/2} \quad \text{and} \quad I(x,t) = i(x,t)Z(x)^{1/2} \tag{2.2}$$

which now have the same dimension. In terms of these normalized variables (2.1) becomes

$$\frac{\partial}{\partial x}\begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix} = \begin{bmatrix} -k(x) & -\dfrac{\partial}{\partial t} \\ -\dfrac{\partial}{\partial t} & k(x) \end{bmatrix}\begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix} \tag{2.3}$$

where $k(x)$ is the *local reflectivity function* (also called sometimes the *local reflection coefficient*) given by

$$k(x) = Z(x)^{-1/2}\frac{d}{dx}Z(x)^{1/2} = \frac{1}{2}\frac{d}{dx}\ln Z(x). \tag{2.4}$$

Note that, as a direct consequence of this normalization, we have

$$\frac{v(x,t)}{i(x,t)} = \frac{V(x,t)}{I(x,t)}Z(x). \tag{2.5}$$

From the system (2.3) we can obtain directly the second order *wave equations*

$$\begin{aligned} &\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)V(x,t) - P(x)V(x,t) = 0 \\ &\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)I(x,t) - Q(x)I(x,t) = 0 \end{aligned} \tag{2.6}$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

where the *potentials* are given by

$$P(x) = -\frac{d}{dx}k(x) + k(x)^2 = Z(x)^{1/2}\frac{d^2}{dx^2}Z(x)^{-1/2},$$
$$Q(x) = \frac{d}{dx}k(x) + k(x)^2 = Z(x)^{-1/2}\frac{d^2}{dx^2}Z(x)^{1/2}. \tag{2.7}$$

In the transform domain the equations (2.6) take the form of Schrödinger equations, which justifies calling $P(x)$ and $Q(x)$ potentials.

From (2.3) we can also obtain a model where the variables of interest are right and left propagating waves defined as

$$W_R(x, t) = \frac{V(x, t) + I(x, t)}{2} \quad \text{and} \quad W_L(x, t) = \frac{V(x, t) - I(x, t)}{2}. \tag{2.8}$$

The evolution of the *wave variables* is given by

$$\frac{\partial}{\partial x}\begin{bmatrix} W_R(x, t) \\ W_L(x, t) \end{bmatrix} = \begin{bmatrix} -\frac{\partial}{\partial t} & -k(x) \\ -k(x) & \frac{\partial}{\partial t} \end{bmatrix}\begin{bmatrix} W_R(x, t) \\ W_L(x, t) \end{bmatrix}. \tag{2.9}$$

To interpret this equation, note that when the impedance is constant over a certain section of the medium we shall have $k(x) = 0$ and therefore $W_R(x, t) = W_R(t - x)$ and $W_L(x, t) = W_L(t + x)$, corresponding to noninteracting right and left propagating waves. The intensity of local interaction between the waves propagating in opposite directions is quantified by $k(x)$, which justifies calling it the local reflectivity. A simple discretization of (2.9) gives the lattice model shown in Fig. 1. Such discrete lattice structures appear in a large number of applications, such as the linear prediction algorithms for speech signals [37], the layered-earth models of Goupillaud [2], [4], [13], and digital filter synthesis [29]. The model in Fig. 1 is in fact crucial to the intuitive understanding of the inverse scattering techniques that we shall derive below.

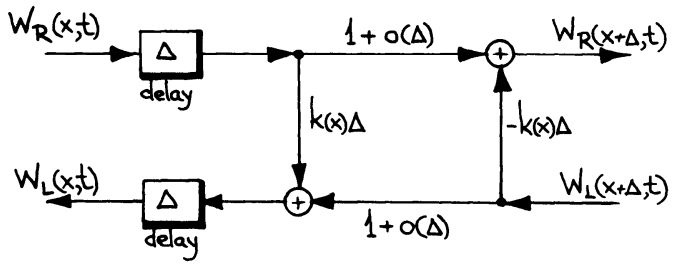

FIG. 1. *Discretized wave scattering layer.*

By performing the space transformation

$$y(x) = \int_0^x Z(\xi)\, d\xi \tag{2.10}$$

on the telegrapher's equation we obtain the *string equation*

$$\frac{\partial^2}{\partial y^2}v(y, t) = \mu(y)\frac{\partial^2}{\partial t^2}v(y, t) \tag{2.11}$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

where $\mu(y)$ is the *mass density of the string* and is given by

$$\mu(y) = Z^{-2}[x(y)] \tag{2.12}$$

where $x(y)$ is the inverse transformation corresponding to (2.10). This model has been used in relating inverse scattering methods to linear estimation theory [35], though the other physical models can be used as well (see [26]–[29]). By using the alternate space transformation

$$y'(x) = \int_0^x Z(\xi)^{-1}\, d\xi, \tag{2.13}$$

we also obtain the *conjugate string equation*

$$\frac{\partial^2}{\partial y^2} i(y', t) = \mu'(y') \frac{\partial^2}{\partial t^2} i(y', t) \tag{2.14}$$

with

$$\mu'(y') = Z^2[x(y')]. \tag{2.15}$$

Note that $\mu[y(x)]\mu'[y'(x)] = 1$, which explains referring to (2.11) and (2.14) as conjugate equations.

The four models of a lossless scattering medium that we use in the sequel are thus the telegrapher's equations (2.1) parametrized by $Z(x)$, the Schrödinger equations (2.6) parametrized by $P(x)$ and $Q(x)$, the two-component wave system (2.9) specified by $k(x)$ and the string equations (2.11) and (2.14) parametrized by $\mu(y)$ and $\mu'(y')$ respectively. The objective of the inverse scattering problem is to reconstruct any of the above parametrizations from some given scattering data. The scattering data are obtained by probing the medium in order to determine its impulse or frequency response at one of the boundaries. The probing signals and the medium response are assumed to be measured perfectly, i.e. *the scattering data will be considered noise free.* Also note that, since $k(x)$ and $P(x)$ and $Q(x)$ are expressed in terms of the first and second derivatives of the impedance function, the different inversion methods will require various degrees of smoothness for $Z(x)$. For the telegrapher's equations (2.1), we shall assume, as was done by Carroll and Santosa [38], that $Z(\cdot) \in C_1[0, \infty)$, or, equivalently that the reflectivity function $k(\cdot) \in L_1[0, \infty)$. As noted by Carroll and Santosa, the smoothness requirements on $Z(\cdot)$ are more stringent for the associated Schrödinger equations (2.6), in which case $Z(\cdot)$ has to belong to $C_2[0, \infty)$ or, equivalently we must have $\int_0^\infty (1+x)|P(x)|\, dx < \infty$ (see also Faddeev [1]).

The inverse scattering problem associated with the Schrödinger equation of quantum physics is complicated by the possible existence of bound states. A consequence of the assumed transmission-line model (2.1), under the above smoothness conditions, is that energy cannot be trapped in the medium, thereby ruling out the possibility of bound states [4], [20], [37].

The four models considered in this section correspond, as stated initially, to lossless scattering media; certain generalizations will be considered in § 6.

3. Continuous parameter Schur recursions. The basic differential inverse scattering method that we discuss in this paper relies on the wave picture associated with equation (2.9), a discrete approximation of which is depicted in Fig. 1.

3.1. The scattering data. The necessary data for the reconstruction of the scattering medium parameters may be obtained in two possible ways.

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

In the first case the medium is assumed to be quiescent at $t=0$ and it is probed by a known rightward propagating waveform incident on the medium after $t=0$. This waveform $W_R(0, t)$ will in general be an impulse followed (in time) by a piecewise continuous function, but we also discuss the case when no leading impulse is present. The measured data is the leftward propagating wave, as it is recorded at $x=0$, $W_L(0, t)$. It can be viewed as obtained by convolving the impulse response $R(t)$, of the scattering medium, with the probing wave $W_R(0, t)$. Since the ultimate objective is to measure the impulse response of the medium, the nature of the probing wave is not important provided it contains enough energy at all frequencies. Note, indeed, that as long as $W_R(0, t)$ is given and $W_L(0, t)$ is measured perfectly, we may always obtain the impulse response by performing a deconvolution.

Another way of gathering scattering data is to perform a measurement of the frequency response $\hat{R}(\omega)$ by sending into the medium sinusoidal waveforms at various frequencies and measuring the magnitude and phase-shift of the returning sinusoidal wave. This is equivalent to the time-domain measurements described above, since $\hat{R}(\omega)$ is the Fourier transform of $R(t)$.

From a practical point of view we cannot always directly generate the waveform $W_R(0, t)$ and measure $W_L(0, t)$. However we usually do have access to the physical variables $v(0, t)$ and $i(0, t)$, and by obtaining the medium response in terms of these variables we can reconstruct the corresponding $W_R(0, t)$ and $W_L(0, t)$ by using (2.8). (In the sequel we assume that $Z(0)=1$.) The nature of the measurements (impulse or frequency response) clearly depends on the physical apparatus that is available. In the geophysical context, approximate impulse responses are obtained by using explosive sources (dynamite, air-guns) and frequency response data can be generated by using wide-band acoustic sources [39].

3.2. The layer peeling procedure. Suppose that the incoming wave $W_R(0, t)$ contains a leading impulse. This impulse will propagate through the medium and, since the medium is causal, it is not hard to recognize by examining Fig. 1, that the waves $W_R(x, t)$ and $W_L(x, t)$ must have the form

$$\begin{aligned} W_R(x, t) &= \delta(t-x) + w_R(x, t)u(t-x), \\ W_L(x, t) &= w_L(x, t)u(t-x) \end{aligned} \tag{3.1}$$

where $w_R(x, t)$ and $w_L(x, t)$ are some piecewise continuous functions, $\delta(\cdot)$ denotes the Dirac distribution, and $u(\cdot)$ is the unit step function, i.e.,

$$u(t) = \begin{cases} 1 & \text{for } t \geqq 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{3.2}$$

The causal nature of $W_R(x, t)$ and $W_L(x, t)$, i.e. that they are zero for $t<x$, is a direct consequence of the fact that the medium was at rest at $t=0$, since the impulse requires an amount of time equal to x to reach the depth x in the medium. Note that we assumed that the perturbation in the medium originated from its left end alone. By substituting (3.1) into the propagating equations (2.9) and equating the coefficients of $\delta(t-x)$ on both sides, we find that

$$w_L(x, t)|_{t=x_+} = w_L(x, x+) = \frac{1}{2}k(x) \quad \text{and} \quad \frac{d}{dx}w_R(x, x+) = -\frac{1}{2}k(x)^2. \tag{3.3}$$

Note that this argument may be regarded as an application of the classical method of *propagation of singularities* (see, e.g. [40]).

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

The above results show that the reflectivity function $k(x)$ can be reconstructed directly from the reflected waves at depth x in the scattering medium. But we have only assumed that the reflected wave at $x=0$ is measured; however, from this information the waves at depth $x>0$ can be constructed by a recursive procedure. Thus let us assume that the waves at point x have already been computed; then $k(x)$ can be readily identified as $w_L(x, x+)$. Using (3.3) and substituting this value into the propagation equations (2.9), we can compute the waves at depth $x+\Delta$.

A simple discretization scheme that can be used to propagate the waves $w_R(x, t)$ and $w_L(x, t)$ is shown in Fig. 2. Note that w_R and w_L are propagated respectively along $\pm 45°$ directions in the (x, t) plane, which correspond to the directions of the characteristics of the hyperbolic system (2.9). Therefore, starting at $x=0$, the resulting recursive algorithm can successively identify the reflectivity function for increasing values of x.

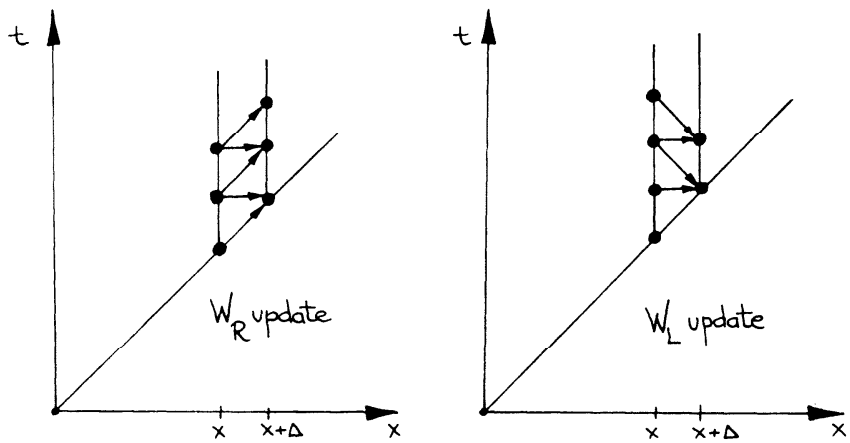

FIG. 2. *Integration path for the propagation of the layer-peeling algorithm.*

This recursive inverse scattering process may be viewed as a *layer-peeling* algorithm, where at every step one infinitesimal layer of the scattering medium is identified and effectively removed. The right and left propagating waves inside the medium are recursively generated and can be regarded at each step as a new set of scattering data for the remaining extent of the medium. For a lossless and discrete layered medium this algorithm is known in geophysics as a *dynamic deconvolution* process [9] and it is called the *downward continuation* method by Bube and Burridge [26]. Dewilde et al. [29], [30], noted that this algorithm is equivalent to the Darlington synthesis procedure for scattering functions and pointed out its similarity to a result of Schur (1917) that will be discussed in the next section. In the context of fast algorithms for linear estimation and operator factorization theory these recursions are sometimes referred to as the *fast Cholesky* recursions [32]. The layer-peeling algorithm described above is also similar in spirit to the medium identification methods obtained by Symes [23] and by Santosa and Schwetlick [24] for the Schrödinger and telegrapher's equations respectively; these methods will be explicitly analyzed in §§ 4.1 and 4.2.

The discretization scheme described in Fig. 2 is only a rough implementation of the layer-peeling procedure, used here only to provide a simple interpretation of that inverse scattering method. More reliable numerical schemes can be obtained by using a method similar to that described in Dewilde, Fokkema and Widya [30]. Denote

$$a_R(x, t) = w_R(x, t+x) \quad \text{and} \quad a_L(x, t) = w_L(x, t+x). \tag{3.4}$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

Then, integrating the evolution equations (2.9), we obtain, after some calculation, the following system of equations

$$
\begin{aligned}
a_R(x, t) &= w_R(0, t) - \int_0^x k(\xi) a_L(\xi, t)\, d\xi, \\
a_L(x, t) &= w_L(0, t+2x) - \int_0^x k(\xi) a_R(\xi, t+2x-2\xi)\, d\xi
\end{aligned}
\tag{3.5}
$$

together with the formula giving the reflectivity function

$$
k(x) = 2a_L(x, 0) = 2\left[w_L(0, 2x) - \int_0^x k(\xi) a_R(\xi, 2x-2\xi)\, d\xi \right]. \tag{3.6}
$$

By recursively integrating (3.5)-(3.6) along successive antidiagonals in the (x, t) plane, using reliable numerical integration schemes, we can obtain the reflectivity function $k(x)$, for increasing values of x. Note also that we only need to know the probing and reflected waves $W_R(0, t)$ and $W_L(0, t)$ over the time span $[0, 2x]$ in order to recover the transmission-line parametrization up to depth x.

From a computational point of view, if we assume that the part of the medium of interest has total length L, and if we use a difference scheme with step-size L/N in the propagation of the layer peeling algorithm, (3.5)-(3.6), the total number of operations required to reconstruct the reflectivity function parametrization is $O(N^2)$. These algorithms are therefore very efficient, when compared to the direct reconstruction methods, formulated in terms of integral/matrix equations; the complexity of these methods will be $O(N^3)$ unless special algorithms are used that exploit the Hankel and/or Toeplitz kernel structure that the physical medium imposes on the integral equations. See also Bube and Burridge [26], for a detailed analysis of the numerical complexity of downward continuation methods.

Up to this point we have only discussed how to reconstruct the reflectivity function $k(\cdot)$. The potentials $P(\cdot)$ and $Q(\cdot)$ can be obtained by combining (2.7) and (3.3). Thus note that the expression (3.1) for $W_R(x, t)$ and $W_L(x, t)$ implies that

$$
\begin{aligned}
V(x, t) &= \delta(t-x) + \Psi(x, t) u(t-x), \\
I(x, t) &= \delta(r-x) + \Phi(x, t) u(t-x)
\end{aligned}
\tag{3.7}
$$

where

$$
\Psi(x, t) = w_R(x, t) + w_L(x, t) \quad \text{and} \quad \Phi(x, t) = w_R(x, t) - w_L(x, t). \tag{3.8}
$$

Therefore by (3.3) and (2.7) the potentials are given by

$$
\begin{aligned}
-2\frac{d}{dx}\Psi(x, x+) &= P(x), \\
-2\frac{d}{dx}\Phi(x, x+) &= Q(x).
\end{aligned}
\tag{3.9}
$$

This reconstruction method requires the existence of $k(x)$ and hence by (2.7) the differentiability of the impedance function $Z(\cdot)$.

3.3. The Schur recursions. By taking the Fourier transform of the waves $W_R(x, t)$ and $W_L(x, t)$ the propagation equations (2.9) become

$$
\frac{d}{dx}\begin{bmatrix} \hat{W}_R(x, \omega) \\ \hat{W}_L(x, \omega) \end{bmatrix} = \begin{bmatrix} -j\omega & -k(x) \\ -k(x) & j\omega \end{bmatrix} \begin{bmatrix} \hat{W}_R(x, t) \\ \hat{W}_L(x, t) \end{bmatrix} \tag{3.10}
$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

and the frequency response, or *reflection coefficient function* of the section of scattering medium over $[x, \infty)$ is given by the ratio

$$\hat{R}(x, \omega) = \frac{\hat{W}_L(x, \omega)}{\hat{W}_R(x, \omega)}. \tag{3.11}$$

Clearly

$$\hat{R}(0, \omega) = \hat{R}(\omega) \tag{3.12}$$

is provided by the given scattering data. Using these definitions the layer-peeling algorithm described above can be recast as a recursive procedure for computing the sequence of reflection coefficient functions $\hat{R}(x, \omega)$ for increasing values of x. Since $\hat{R}(x, \omega)$ is the ratio of variables with a linear evolution given by (3.10), it will not be surprising to find, after some algebra, that it satisfies the Riccati equation

$$\frac{d}{dx}\hat{R}(x, \omega) = 2j\omega\hat{R}(x, \omega) + k(x)[\hat{R}(x, \omega)^2 - 1]. \tag{3.13}$$

It is not clear how this can help, since $k(\cdot)$ is unknown, but recalling the identity (3.3) for $k(x)$ and the form (3.1) for the waves at x, we find by using the initial value theorem for unilateral transforms that

$$k(x) = 2w_L(x, x+) = \lim_{\omega\to\infty} 2j\omega\hat{R}(x, \omega). \tag{3.14}$$

In terms of the causal impulse response $R(x, t)$ corresponding to the reflection function $\hat{R}(x, \omega)$, the equation (3.14) simply states that

$$k(x) = 2R(x, 0+). \tag{3.15}$$

The Riccati equation (3.13) for the reflection function is fairly well-known in radiative transfer and transmission-line theory, and is a direct consequence of the rules of cascading infinitesimal scattering layers [41]. More details about the evolution of the medium representation under successive compositions of infinitesimal scattering layers will be given in § 5.

In the context of the inverse problem of geophysics the Riccati equation (3.13) was also obtained by Gjevik et al. [42]. However, they did not notice the relation (3.14) which enables to propagate the Riccati equation recursively (in an autonomous manner) starting from the scattering data $\hat{R}(0, \omega) = \hat{R}(\omega)$. They proposed an iterative, rather than a recursive procedure to compute the $k(x)$ function. We note at this point that the propagation of equations (3.13) and (3.14) may not provide a numerically reliable way to reconstruct the reflectivity function $k(\cdot)$, especially if we compare it to the recursions described in the previous section. However, in the discrete parameter case, a similar scheme was used by Pekeris, [43]–[44], for the study of an inverse resistivity problem. Therefore, further work on obtaining stable implementations of a Riccati equation-based inversion procedure may be worth pursuing.

The equations (3.13) and (3.14) constitute the continuous version of a procedure derived by Schur, in 1917 [27], [28], for testing the boundedness of an analytic function outside the unit circle of the complex plane. Given a power series in z^{-1}, $S_0(z)$, Schur proved that $|S_0(z)| < 1$ for $|z| \geqq 1$ if and only if the sequence of coefficients k_n generated by the recursion

$$S_{n+1}(z) = z\frac{S_n(z) - k_n}{1 - k_n S_n(z)} \quad \text{with } k_n = \lim_{z\to\infty} S_n(z) \tag{3.16}$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

are such that $|k_n| \leq 1$. The recursion (3.16) is in fact a discretized form of the Riccati equation (3.13), and can be obtained from it by using a backwards difference scheme.

The Schur algorithm (3.16) may be interpreted as testing for the existence of a discrete (i.e. with piecewise constant impedance) transmission-line having $S_0(z)$ for the left reflection coefficient function. Similarly, the continuous version of this algorithm may be considered as testing for the existence of a lossless transmission-line for realizing the given scattering function $\hat{R}(\omega)$. A condition for the existence of such a transmission-line is that the reconstructed local impedance function $Z(x)$, appearing in the model (2.1), should be strictly positive and bounded. Since, from (2.4)

$$Z(x) = Z(0) \exp 2\left\{\int_0^x k(\xi)\, d\xi\right\} \tag{3.17}$$

this implies that we need to have $|\int_0^x k(\xi)\, d\xi| < \infty$ for all x, a condition we already assumed (see § 2). In this case the scattering function $\hat{R}(\omega)$ is bounded by 1 in the lower half plane. (Because our sign convention for the Fourier transform, $\hat{R}(\omega)$, is analytic in the lower half plane instead of the upper half plane as in most references on mathematical inverse scattering theory.) We note that if a transmission-line is lossless, its left reflection function $\hat{R}(\omega)$ must be bounded by one on the real axis as a result of energy conservation [2], [5], [7], [18].

4. Differential inversion methods for $Z(\cdot)$ and $\{P(\cdot), Q(\cdot)\}$. In the previous section our analysis concentrated on the two-component system of wave equations (2.9), and in this framework we have shown how to reconstruct the local reflectivity function $k(\cdot)$ and, via the identities (2.7) and (3.17), also the potentials $P(\cdot)$ and $Q(\cdot)$, and the impedance $Z(\cdot)$. However, because $k(\cdot)$ is expressed as a function of the first derivative of the local impedance function, the reconstruction method of § 3 requires the differentiability of $Z(\cdot)$. When the local impedance function is only piecewise differentiable, the Schur algorithm can be modified to take the discontinuities into account. However a more direct method is to use an impedance reconstruction method, which can be described as follows.

4.1. The impedance reconstruction method. Assume that the probing wave $W_R(x, t)$ does *not* contain a leading impulse and is a piecewise continuous function starting at $t = 0$. Then, by causality, $W_R(x, t)$ and $W_L(x, t)$ must have the form

$$\begin{aligned} W_R(x, t) &= w_R(x, t) u(t - x), \\ W_L(x, t) &= w_L(x, t) u(t - x). \end{aligned} \tag{4.1}$$

Substituting these expressions into (2.9), we find that

$$w_L(x, x+) = 0 \tag{4.2}$$

which implies that

$$V(x, x+) = I(x, x+). \tag{4.3}$$

Recalling the identity (2.5), this shows that we have

$$\frac{v(x, x+)}{i(x, x+)} = Z(x). \tag{4.4}$$

Therefore, to reconstruct the impedance function, $Z(\cdot)$, we only need to measure the voltage and current variables $v(0, t)$ and $i(0, t)$ at the left boundary of the scattering medium and to propagate $v(x, t)$ and $i(x, t)$ by using (4.4) and (2.1). Note that the

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

knowledge of the voltage and current variables at depth x enables us to compute the impedance $Z(x)$, which in turn can be used to obtain the functions $v(x+\Delta, t)$ and $i(x+\Delta, t)$. In this manner the impedance $Z(x)$ is computed recursively, starting from $x=0$. This procedure was first derived by Santosa and Schwetlick [24] (see also Sondhi and Resnick [25]) who called it the "method of characteristics". However, since the layer-peeling method described in the previous section could also be viewed as a method of characteristics, we shall call this procedure the *impedance reconstruction method.*

The impedance reconstruction procedure can be interpreted in terms of the layer-peeling technique of § 3.2 by considering the discretized version of (2.1) shown in Fig. 3. This figure indicates that the current and voltage variables at point $(n+1)\Delta$, where Δ is the discretization step-size, are obtained from the corresponding variables at depth $n\Delta$ by cascading a scattering layer described by the matrix

$$\Sigma_n = \begin{bmatrix} Z(n\Delta)^{1/2} & 1 \\ Z(n\Delta) & 2Z(n\Delta)^{1/2} \end{bmatrix} \tag{4.5}$$

with time delays and the inverse of the first scattering layer. This result can be obtained by noting that

$$\begin{bmatrix} W_R(n\Delta, t) \\ W_L(n\Delta, t) \end{bmatrix} = \Theta_n \begin{bmatrix} i(n\Delta, t) \\ v(n\Delta, t) \end{bmatrix} \tag{4.6}$$

where

$$\Theta_n = \frac{1}{2}\begin{bmatrix} Z(n\Delta)^{1/2} & Z(n\Delta)^{-1/2} \\ -Z(n\Delta)^{1/2} & Z(n\Delta)^{-1/2} \end{bmatrix} = [P_+\Sigma_n + P_-][P_-\Sigma_n + P_+]^{-1} \tag{4.7}$$

is the chain scattering or transmission matrix corresponding to the scattering representation Σ_n [45]. The projection matrices P_+ and P_- appearing in the above formula are defined as follows

$$P_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad P_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.8}$$

The method of characteristics exploits the delay structure as displayed in Fig. 3 and the fact that the left reflection coefficient (i.e. the 21 entry) of the matrix Σ_n is the local impedance $Z(n\Delta)$. Since both Σ_n and its inverse are entirely parametrized by the local impedance, the scattering layers associated to these matrices can be easily "peeled off" (i.e. their effect may be accounted for) as soon as $Z(n\Delta)$ has been computed.

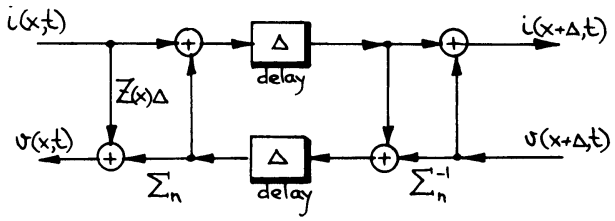

FIG. 3. *Discretized medium associated with the impedance reconstruction procedure.*

The identity (4.4) shows that the reconstruction procedure described above can also be used to obtain the mass densities $\mu(\cdot)$ and $\mu'(\cdot)$, appearing in the string

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

equations (2.11) and (2.14). This is done by substituting

$$\mu(y)=\left[\frac{i(y,y+)}{v(y,y+)}\right]^2 \tag{4.9}$$

and

$$\mu'(y')=\left[\frac{v(y',y'+)}{i(y',y'+)}\right]^2 \tag{4.10}$$

into the equations

$$\frac{\partial}{\partial y}\begin{bmatrix} v(y,t) \\ i(y,t) \end{bmatrix}=\begin{bmatrix} 0 & -\dfrac{\partial}{\partial t} \\ -\mu(y)\dfrac{\partial}{\partial t} & 0 \end{bmatrix}\begin{bmatrix} v(y,t) \\ i(y,t) \end{bmatrix} \tag{4.11}$$

and

$$\frac{\partial}{\partial y'}\begin{bmatrix} v(x,y') \\ i(y',t) \end{bmatrix}=\begin{bmatrix} 0 & -\mu(y')\dfrac{\partial}{\partial t} \\ -\dfrac{\partial}{\partial t} & 0 \end{bmatrix}\begin{bmatrix} v(y',t) \\ i(y',t) \end{bmatrix} \tag{4.12}$$

that describe the strings (2.11) and (2.14).

4.2. Direct recovery of the potential. Similarly, there also exists a procedure for computing the potentials $P(\cdot)$ and $Q(\cdot)$ directly, without first reconstructing the reflectivity function $k(\cdot)$. To do so let

$$\begin{aligned} F(x,t)&=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)V(x,t),\\ G(x,t)&=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)I(x,t). \end{aligned} \tag{4.13}$$

Then the Schrödinger equations (2.6) can be rewritten in the form of asymmetric two-component differential systems, given by

$$\frac{\partial}{\partial x}\begin{bmatrix} V(x,t) \\ F(x,t) \end{bmatrix}=\begin{bmatrix} -\dfrac{\partial}{\partial t} & 1 \\ P(x) & \dfrac{\partial}{\partial t} \end{bmatrix}\begin{bmatrix} V(x,t) \\ F(x,t) \end{bmatrix} \tag{4.14}$$

and

$$\frac{\partial}{\partial x}\begin{bmatrix} I(x,t) \\ G(x,t) \end{bmatrix}=\begin{bmatrix} -\dfrac{\partial}{\partial t} & 1 \\ Q(x) & \dfrac{\partial}{\partial t} \end{bmatrix}\begin{bmatrix} I(x,t) \\ G(x,t) \end{bmatrix}. \tag{4.15}$$

The layer-peeling technique introduced in § 3 can again be used to recover the potentials $P(\cdot)$ and $Q(\cdot)$ directly, by noting that

$$\begin{aligned} P(x)&=-2F(x,x+),\\ Q(x)&=-2G(x,x+). \end{aligned} \tag{4.16}$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

Consequently, if we propagate the variables $\{V(x, t), F(x, t)\}$ or $\{I(x, t), G(x, t)\}$ by using (4.16) and the propagation equations (4.14)-(4.15), the potentials $P(\cdot)$ and $Q(\cdot)$ can be recovered directly from the scattering data.

To obtain the initial conditions for the systems (4.14) and (4.15), we asume that the scattering data is $W_R(0, t) = \delta(t)$ and $W_R(0, t) = R(t)u(t)$. Then, by using equation (2.3) and the fact that $k(0) = 2R(0+)$, we find that

$$\begin{aligned} V(0, t) &= \delta(t) + R(t)u(t), \\ F(0, t) &= -2\left[-\frac{d}{dt}R(t) + R(0+)R(t)\right]u(t) \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} I(0, t) &= \delta(t) - R(t)u(t), \\ G(0, t) &= -2\left[\frac{d}{dt}R(t) + R(0+)R(t)\right]u(t). \end{aligned} \tag{4.18}$$

Whereas in § 3 the potential was reconstructed by using the original scattering data and then differentiating the reflectivity function, the method that we propose here first differentiates the scattering data and then reconstructs the potential directly. An alternative differential potential reconstruction method, which can also be interpreted as a layer stripping process, is analyzed by Symes [23].

The layer-peeling algorithm for the systems (4.14), (4.15) can be interpreted as successively truncating the potentials $P(\cdot)$ and $Q(\cdot)$ in such a way that the new potentials

$$\begin{aligned} P(z, x) &= P(z)u(z-x), \\ Q(z, x) &= Q(z)u(z-x) \end{aligned} \tag{4.19}$$

correspond to the part of the original scattering medium located to the right of x. In this interpretation it is assumed that the part of the scattering medium on the left of x that was removed by the layer-peeling algorithm has been replaced by free-space (i.e. $k(z) = 0$ for $z < x$). The idea of using truncated potentials for the analysis of direct scattering phenomena was exploited earlier by Bellman and Wing [46] and is discussed in Lamb [47]. This approach may also be regarded as an invariant imbedding method.

The differential method presented above for the reconstruction of the potentials $P(\cdot)$ and $Q(\cdot)$ seems to be related to the *trace method* of Deift and Trubowitz. Their method is based on the recursive computation of the Jost solution of the Schrödinger equation given by

$$\frac{d^2}{dx^2}f(x, \omega) + [\omega^2 - P(x)]f(x, \omega) = 0 \tag{4.20}$$

with boundary condition

$$\lim_{x\to\infty} f(x, \omega) \exp\{-j\omega x\} = 1. \tag{4.21}$$

Then, by substituting the trace formula

$$P(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} j\omega\hat{R}(\omega)f^2(x, \omega)\, d\omega \tag{4.22}$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

into (4.20), $f(x, \omega)$ and $P(x)$ can be computed recursively for decreasing values of x. The connection between the approach of Deift and Trubowitz, and the algorithm that we have discussed above is not yet completely understood.

5. Integral equations for inverse scattering. In § 3 the Schur recursions were derived only using causality and the differential description of the medium. This is in contrast to most classical inversion methods which are formulated in terms of integral equations. In this section we shall show how such equations arise as a simple consequence of the symmetry, losslessness and causality properties of the scattering medium described by (2.9). A variety of special integral equations (e.g. Marchenko, Gelfand–Levitan, Krein) can be obtained by considering particular input-output pairs $\{W_R(0, t), W_L(0, t)\}$. In § 5.2 we shall present an integral equation for general scattering data and show how its structure can be exploited to obtain fast solution algorithms. Then we show how appropriate choices of $\{W_R(0, t), W_L(0, t)\}$ lead to all the classical equations of inverse scattering theory.

5.1. Transmission and scattering descriptions of the medium. The system (2.9) describes the transmission of waves through an infinitesimal section of the medium. These infinitesimal layers may be aggregated over the interval $[0, x]$ and by using the linearity of the medium we find that the waves $W_R(x, t)$ and $W_L(x, t)$ at depth x are related to the waves at the boundary by

$$\begin{bmatrix} W_R(x, t) \\ W_L(x, t) \end{bmatrix} = \begin{bmatrix} M_{11}(x, t) & M_{12}(x, t) \\ M_{21}(x, t) & M_{22}(x, t) \end{bmatrix} * \begin{bmatrix} W_R(0, t) \\ W_L(0, t) \end{bmatrix} \tag{5.1}$$

where $*$ denotes the convolution operator. The matrix

$$M(x, t) = \begin{bmatrix} M_{11}(x, t) & M_{12}(x, t) \\ M_{21}(x, t) & M_{22}(x, t) \end{bmatrix} \tag{5.2}$$

is the transition matrix of the medium over $[0, x]$ and it satisfies the differential equation

$$\frac{\partial}{\partial x} M(x, t) = \begin{bmatrix} -\dfrac{\partial}{\partial t} & -k(x) \\ -k(x) & \dfrac{\partial}{\partial t} \end{bmatrix} M(x, t) \tag{5.3}$$

with initial condition

$$M(0, t) = \begin{bmatrix} \delta(t) & 0 \\ 0 & \delta(t) \end{bmatrix}. \tag{5.4}$$

The aggregated medium corresponding to $M(x, t)$ can be viewed as obtained by composing the infinitesimal layers that were peeled off from the medium by the Schur algorithm over the interval $[0, x]$. Let $\hat{M}(x, \omega)$ be the Fourier transform of $M(x, t)$. Then, the composition procedure for generating $\hat{M}(x, \omega)$ and the layer peeling method are compared in Fig. 4.

Instead of using the transmission description of the medium given by (5.1) we can use an equivalent scattering description that relates the outgoing waves to the incoming waves as follows

$$\begin{bmatrix} W_R(x, t) \\ W_L(0, t) \end{bmatrix} = \begin{bmatrix} T_L(x, t) & R_R(x, t) \\ R_L(x, t) & T_R(x, t) \end{bmatrix} * \begin{bmatrix} W_R(0, t) \\ W_L(x, t) \end{bmatrix}. \tag{5.5}$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

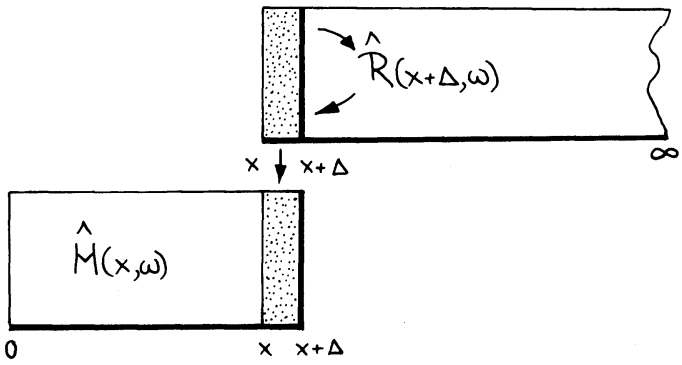

FIG. 4. *Comparison of the layer-peeling and layer aggregation methods.*

The Fourier transform $\hat{S}(x, \omega)$ of the matrix

$$S(x, t) = \begin{bmatrix} T_L(x, t) & R_R(x, t) \\ R_L(x, t) & T_R(x, t) \end{bmatrix} \tag{5.6}$$

is the *scattering matrix* associated to the medium over $[0, x]$ and it can be obtained from $\hat{M}(x, \omega)$ by the relation

$$\hat{S}(x, \omega) = [P_+\hat{M}(x, \omega) + P_-][P_-\hat{M}(x, \omega) + P_+]^{-1}. \tag{5.7}$$

where $P_\pm$ are the projection operators defined by (4.8). The general rules of composition of scattering layers are described in Redheffer [41], though we shall not need them here.

As a consequence of the delay structure and losslessness of the elementary (infinitesimal) scattering layers described in Fig. 1, the scattering matrix $S(x, t)$ is such that

$$R_R(x, t) = R_L(x, t) = 0 \quad \text{for } t < 0, \tag{5.8}$$

$$T_R(x, t) = T_L(x, t) = 0 \quad \text{for } t < x \tag{5.9}$$

and it is lossless, i.e.

$$\hat{S}^H(x, \omega)\hat{S}(x, \omega) = I. \tag{5.10}$$

where the superscript H denotes Hermitian transpose. In the transmission representation domain, the relations (5.8) and (5.9) imply that the entries of $M(x, \cdot)$ have all support over $[-x, x]$. Finally, by noting that the transmission medium is invariant when the right and left propagating waves are interchanged and time is reversed, we get the following useful identities

$$\begin{aligned} M_{11}(x, t) &= M_{22}(x, -t), \\ M_{21}(x, t) &= M_{12}(x, -t). \end{aligned} \tag{5.11}$$

5.2. General integral equations. When the medium is probed from the left, a consequence of its delay structure is that

$$W_R(x, t) = W_L(x, t) = 0 \quad \text{for } t < x. \tag{5.12}$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

By substituting (5.12) into (5.1) and recalling that $M(x, \cdot)$ has support on $[-x, x]$, we obtain the system of integral equations

$$\begin{aligned} &\int_{-x}^{t} W_R(0, t-\tau) M_{11}(x, \tau)\, d\tau + \int_{-x}^{t} W_L(0, t-\tau) M_{12}(x, \tau)\, d\tau = 0, \\ &\int_{-x}^{t} W_R(0, t-\tau) M_{21}(x, \tau)\, d\tau + \int_{-x}^{t} W_L(0, t-\tau) M_{22}(x, \tau)\, d\tau = 0 \end{aligned} \tag{5.13}$$

which relates the entries of $M(x, t)$ to the measured waves $W_R(0, t)$ and $W_L(0, t)$, i.e. the scattering data. From the differential equation (5.3)-(5.4) satisfied by $M(x, t)$, it can be shown that $M_{11}(x, t)$ and $M_{21}(x, t)$ can be expressed as

$$\begin{aligned} M_{11}(x, t) &= \delta(x-t) + m_{11}(x, t)[u(t+x) - u(t-x)], \\ M_{21}(x, t) &= m_{21}(x, t)[u(t+x) - u(t-x)]. \end{aligned} \tag{5.14}$$

Then, by using (5.14) and the symmetry relations (5.11), we get the integral equations (for $-x \leqq t \leqq x$)

$$\begin{aligned} &\int_{-x}^{t} W_R(0, t-\tau) m_{11}(x, \tau)\, d\tau + \int_{-t}^{x} W_L(0, t+\tau) m_{21}(x, \tau)\, d\tau = 0, \\ &W_L(0, t+x) + \int_{-t}^{x} W_L(0, t+\tau) m_{11}(x, \tau)\, d\tau + \int_{-x}^{t} W_R(0, t-\tau) m_{21}(x, \tau)\, d\tau = 0, \end{aligned} \tag{5.15}$$

which need to be solved for the functions $m_{11}(x, \cdot)$ and $m_{21}(x, \cdot)$. To guarantee the existence of solutions to these equations, it is as usual assumed that the probing wave $W_R(0, t)$ contains a leading impulse, see e.g. (3.1). In this case the integral equations (5.15) take the form of a system of coupled Fredholm equations of the second kind

$$\begin{aligned} &m_{11}(x, t) + \int_{-x}^{t} w_R(0, t-\tau) m_{11}(x, \tau)\, d\tau + \int_{-t}^{x} w_L(0, t+\tau) m_{21}(x, \tau)\, d\tau = 0, \\ &w_L(0, t+x) + m_{21}(x, t) + \int_{-t}^{x} w_L(0, t+\tau) m_{11}(x, \tau)\, d\tau \\ &\qquad\qquad + \int_{-x}^{t} w_R(0, t-\tau) m_{21}(x, \tau)\, d\tau = 0. \end{aligned} \tag{5.16}$$

The solution of these equations can be used to reconstruct the medium since from (5.3) and exploiting the form (5.14) of $M_{11}(x, t)$ and $M_{21}(x, t)$ we find that

$$k(x) = -2m_{21}(x, x-) \tag{5.17}$$

and

$$k^2(x) = 2\frac{d}{dx} m_{11}(x, x-). \tag{5.18}$$

The equations (5.16) can be solved directly by using a simple discretization scheme. If the interval $[-x, x]$ is divided into N equal subintervals, this scheme would require $O(N^3)$ operations in order to reconstruct the reflectivity function over $[0, x)$.

However the kernels $w_R(0, t-\tau)$ and $w_L(0, t+\tau)$ which appear in (5.16) have respectively a Toeplitz and Hankel structure which can be exploited to reduce the number of computations. Thus, note that $m_{11}(x, t)$ and $m_{12}(x, t)$ satisfy the differential

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

system

$$\frac{\partial}{\partial x}\begin{bmatrix} m_{11}(x,t) \\ m_{21}(x,t) \end{bmatrix} = \begin{bmatrix} -\frac{\partial}{\partial t} & -k(x) \\ -k(x) & \frac{\partial}{\partial t} \end{bmatrix} \begin{bmatrix} m_{11}(x,t) \\ m_{21}(x,t) \end{bmatrix} \tag{5.19}$$

for $-x \leqq t \leqq x$, with initial conditions

$$m_{11}(0,0) = m_{21}(0,0) = 0. \tag{5.20}$$

When propagating (5.19), it turns out that we need to supply the values of the kernel $m_{21}(x,\cdot)$ at $t = x-$ (providing $k(x)$) and also the value of $m_{11}(x,\cdot)$ at $t = -x$. By using the second equation of (5.16), $m_{21}(x, x-)$ can be expressed as

$$\begin{aligned} m_{21}(x,x-) = -w_L(0,2x) &- \int_{-x}^{x} w_L(0, x+\tau) m_{11}(x,\tau)\, d\tau \\ &- \int_{-x}^{x} w_R(0, x-\tau) m_{21}(x,\tau)\, d\tau. \end{aligned} \tag{5.21}$$

Furthermore setting $t = -x$ in (5.16), we find that

$$m_{11}(x,-x) = 0. \tag{5.22}$$

The differential system (5.19), with the boundary conditions (5.21) and (5.22), can now be used to compute $m_{11}(x,t)$ and $m_{21}(x,t)$ recursively. The equations (5.19) have the same form as the Schur recursions that were derived in § 3. However the Schur algorithm is formulated as an initial value problem whereas the recursions derived above rely on inner product computations, i.e. (5.21), to determine certain boundary values needed for propagation. These latter recursions are similar to the *Krein-Levinson* equations for factoring the resolvent of a Toeplitz kernel [31], [32]. They require the same order of computations as the Schur recursions, when serial computation is used, but since the Schur recursions avoid inner products, they are more efficient for parallel computation.

The differential equations (5.19) could have been derived also by applying the *displacement operators*

$$r = \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \quad \text{and} \quad \theta = \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \tag{5.23}$$

to the integral equations (5.16), and by using the displacement property

$$\tau f(x-t) = 0 \quad \text{and} \quad \theta f(x+t) = 0 \tag{5.24}$$

of Toeplitz and Hankel kernels. These displacement properties have been exploited in [31], [32] and [46] to derive some fast algorithms for the computation of resolvents of Toeplitz and Hankel operators and to obtain triangular factorizations of these operators. A more general treatment, and stability analysis of the resulting algorithms, has recently been given by Gohberg and Koltracht [34].

5.3. Relation to classical inverse scattering. The integral equations (5.16) are expressed in terms of the general scattering data $W_R(0,t) = \delta(t) + w_R(0,t)u(t)$ and $W_L(0,t) = w_L(0,t)u(t)$. In the literature, two choices for the probing waves have been

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

made either directly or implicitly

(5.25) (i) $w_R(0, t) = 0$ and $w_L(0, t) = R(t)u(t)$,

(5.26) (ii) $w_R(0, t) = w_L(0, t) = h(t)$.

The second choice above arises naturally when the scattering medium is terminated at its left boundary with a perfect reflector. The integral equations that we have obtained above are therefore more general than those presented in the literature of two-component inverse scattering problems [20]. Furthermore they can be used to obtain the classical integral equations that solve the inverse scattering problem for the one-dimensional Schrödinger equation. To do so, denote by

$$K(x, t) = m_{11}(x, t) + m_{21}(x, t). \tag{5.27}$$

Then, by adding the two integral equations (5.16), we obtain

$$w_L(0, t+x) + K(x, t) + \int_{-x}^{x} [w_R(t-\tau) + w_L(t+\tau)]K(x, \tau)\, d\tau = 0 \tag{5.28}$$

where the potential is given by

$$P(x) = 2\frac{d}{dx}K(x, x-). \tag{5.29}$$

In the special case when the scattering data is given by (5.25), the above equations correspond to the "classical" Marchenko solution of the inverse scattering problem [2], [18], [40]. When (5.26) is given as scattering data, it can be shown from (5.28) that the symmetrized kernel

$$K_S(x, t) = \tfrac{1}{2}[K(x, t) + K(x, -t)] \tag{5.30}$$

satisfies the Gelfand–Levitan equation [3], [40],

$$\begin{aligned} &K_S(x, t) + \tfrac{1}{2}[h(x+t) + h(x-t)] \\ &\quad + \int_0^x \tfrac{1}{2}[h(|t-\tau|) + h(|t+\tau|)]K_S(x, \tau)\, d\tau = 0, \qquad 0 < t < x \end{aligned} \tag{5.31}$$

and again

$$P(z) = 2\frac{d}{dx}K_S(x, x+). \tag{5.32}$$

Note that the symmetric kernel $K_S(x, t)$ is half the sum of all the entries in the transmission matrix $M(x, t)$.

Finally, if we define

$$L(x, t) = m_{11}(x, t) + m_{12}(x, t) \tag{5.33}$$

and use the scattering data (5.26), replacing t by $-t$ in the second equation in (5.16) and adding it to the first equation, we find that

$$h(x-t) + L(x, t) + \int_{-x}^{x} h(|t-\tau|)L(x, \tau)\, d\tau = 0. \tag{5.34}$$

This result is known as the Krein integral equation [1], [2], [16], and we have immediately that $K_S(x, t) = L(x, t) + L(x, -t)$. By noting that $m_{11}(x, -x) = 0$ and that $2m_{12}(x, -x) = k(x)$ we find that

$$k(x) = -2L(x, -x) \tag{5.35}$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

so that the local reflectivity function, and therefore the potential, can be reconstructed by this method.

This development shows that all the classical solutions of the inverse scattering problem based on integral equations can be related to the differential approach that we have described in the previous sections. The integral equations based method of Gopinath and Sondhi [6], [7] may be regarded as using a special approach to the solution of the Krein integral equation. This method is of importance since the local impedance is directly recovered and a discussion of it can be found in Bruckstein and Kailath [48].

The classical integral-equations-based inversion methods are also reviewed in Burridge, [40], through an approach involving the derivation of Green's functions, which is similar to the one we used in this section. However, the difference between the two approaches is that we have taken a system-theoretic point of view for the composition of the transmission layers associated with the scattering medium. This enabled us to analyze the integral-equation-based methods simultaneously, and obtain the general equation (5.16), whereas Burridge had to separately consider the Green functions associated to each inversion method.

6. Inverse scattering for general media. The differential inversion methods that we have obtained above were restricted to the case of lossless scattering media. However, it is possible to extend them to more general media, where the wave propagation is described by the two-component system

$$\frac{\partial}{\partial x}\begin{bmatrix} W_R(x,t) \\ W_L(x,t) \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial}{\partial t}+a(x) & \beta(x) \\ b(x) & \dfrac{\partial}{\partial t}+\alpha(x) \end{bmatrix} \begin{bmatrix} W_R(x,t) \\ W_L(x,t) \end{bmatrix}. \tag{6.1}$$

Such systems appear for example in the study of lossy transmission-lines and acoustic media [25], [49]. The local loss function for this system is given by

$$\begin{aligned} &-\frac{d}{dx}\int_{-\infty}^{\infty} [|\hat{W}_R(x,\omega)|^2 - |\hat{W}_L(x,\omega)|^2]\, d\omega \\ &= \int_{-\infty}^{\infty} [W_R'(x,\omega)\, W_L'(x,\omega)] \begin{bmatrix} -2a(x) & b(x)-\beta(x) \\ b(x)-\beta(x) & 2\alpha(x) \end{bmatrix} \begin{bmatrix} W_R(x,\omega) \\ W_L(x,\omega) \end{bmatrix} d\omega \end{aligned} \tag{6.2}$$

which shows that a necessary and sufficient condition for losslessness is that, for all x,

$$\begin{aligned} a(x) &= \alpha(x) = 0, \\ b(x) &= \beta(x). \end{aligned} \tag{6.3}$$

This is the case that was considered in the previous sections. An infinitesimal layer of the scattering medium corresponding to (6.1) is depicted in Fig. 5.

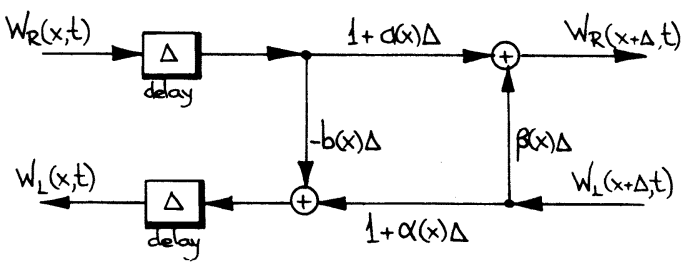

FIG. 5. *Wave scattering picture for a general medium.*

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

Since the scattering medium is parametrized by four different functions $\{a(\cdot), b(\cdot), \alpha(\cdot), \beta(\cdot)\}$, in general it will not be possible to reconstruct all of them from the pair of waves $W_R(0, t)$ and $W_L(0, t)$. The reconstruction techniques that will be discussed in this section therefore assume either that the parameters $a(x)$, $\alpha(x)$ and $\beta(x)$ can be expressed as some function of $b(x)$, or that we have more information than the medium impulse response at $x=0$. Unlike in some earlier sections, the results presented in this section are entirely new.

6.1. Reconstruction for a medium parametrized by $b(x)$ only. When the medium is entirely parametrized in terms of $b(\cdot)$—as was the case, for example, for lossless media—the layer peeling procedure of § 3 can be extended [48]. To do so, note that when the medium is probed by a wave $W_R(0, t)$ with a leading impulse, the waves at some depth x are of the form

$$\begin{aligned} W_R(x, t) &= \gamma_R(x)\delta(t-x) + w_R(x, t)u(t-x), \\ W_L(x, t) &= w_L(x, t)u(t-x) \end{aligned} \tag{6.4}$$

with

$$\gamma_R(x) = \exp\left\{\int_0^x a(\xi)\, d\xi\right\}. \tag{6.5}$$

In this case

$$b(x) = 2\gamma_R^{-1}(x) w_L(x, x+) \tag{6.6}$$

and, since $\gamma_R(x)$ can be obtained from the previously reconstructed layers, equation (6.6) may be used to compute $b(x)$ for the next infinitesimal layer, which in turn determines $a(x)$, $\alpha(x)$ and $\beta(x)$. This implies that (6.1) and (6.6) can recursively compute the waves that propagate inside the medium and simultaneously recover the medium parameters.

The requirement that the medium be parametrized by $b(\cdot)$ alone might seem rather strong; however, in the literature one often encounters papers that, after stating the problem in its full generality, introduce an equivalent assumption. It is also interesting to note that, in case the parameters $a(x)$, $\alpha(x)$ and $\beta(x)$ depend on $b(x)$ in a nontrivial way, it is not clear how the integral equations based inversion approaches can be extended.

6.2. Inverse scattering for a nonsymmetric system. Another example for which differential reconstruction methods can be devised is when $a(x) = \alpha(x) = 0$ in (6.1). In this case the resulting asymmetric two-component system is of the type considered by Zakharov and Shabat (see, e.g. [19], [20]). The system (6.1) can in fact always be reduced to this particular form by performing the substitution

$$\begin{aligned} W_R(x, t) &\leftarrow \gamma_R^{-1}(x) W_R(x, t), \\ W_L(x, t) &\leftarrow \gamma_L^{-1}(x) W_L(x, t) \end{aligned} \tag{6.7}$$

where $\gamma_R(x)$ is given by (6.5) and

$$\gamma_L(x) = \exp\left\{\int_0^x \alpha(\xi)\, d\xi\right\}. \tag{6.8}$$

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

In terms of these "normalized" variables, (6.1) becomes

$$\frac{\partial}{\partial x}\begin{bmatrix} W_R(x,t) \\ W_L(x,t) \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial}{\partial t} & \beta(x)\dfrac{\gamma_L(x)}{\gamma_R(x)} \\ b(x)\dfrac{\gamma_R(x)}{\gamma_L(x)} & \dfrac{\partial}{\partial t} \end{bmatrix}\begin{bmatrix} W_R(x,t) \\ W_L(x,t) \end{bmatrix} \tag{6.9}$$

which is now in the form of an asymmetric two-component system. Let us define

$$k(x) = -b(x)\frac{\gamma_R(x)}{\gamma_L(x)} \quad \text{and} \quad k^A(x) = -\beta(x)\frac{\gamma_L(x)}{\gamma_R(x)}. \tag{6.10}$$

The generalized Schur procedure that we derive next reconstructs the two functions $k(\cdot)$ and $k^A(\cdot)$ which are two independent functions of the original parametrization. Thus, unless $a(x) = \alpha(x) = 0$ for all x, this method will provide only a partial reconstruction of the original medium. Our presentation follows that of Yagle and Levy [50].

In addition to the causal pair of waves $W_R(0, t)$ and $W_L(0, t)$ that was used earlier as scattering data, it will be assumed that we are also given a *noncausal* wave pair

$$\begin{aligned} W_L^A(0,t) &= \delta(t) + w_L^A(0,t)u(-t), \\ W_R^A(0,t) &= w_R^A(0,t)u(-t). \end{aligned} \tag{6.11}$$

These waves can be viewed as obtained by exchanging the role of $W_R(\cdot)$ and $W_L(\cdot)$ and by reversing time in a scattering experiment. The corresponding reflection coefficient function

$$\hat{R}^A(\omega) = \frac{\hat{W}_R^A(0,\omega)}{\hat{W}_L^A(0,\omega)} \tag{6.12}$$

is the $(1, 2)$ entry of $\hat{S}^{-1}(\omega)$, where

$$\hat{S}(\omega) = \begin{bmatrix} \hat{T}_R(\omega) & \hat{R}_L(\omega) \\ \hat{R}_R(\omega) & \hat{T}_L(\omega) \end{bmatrix} \tag{6.13}$$

is the scattering matrix associated with the medium over $[0, \infty)$. It can therefore be obtained by proving the medium from both ends and measuring all the entries of $\hat{S}(\omega)$. Thus, even though the knowledge of the noncausal waves $W_L^A(0, t)$ and $W_R^A(0, t)$ is nonphysical, it can be assumed that $\hat{R}^A(\omega)$ or its anticausal inverse Fourier transform $R^A(t)$ is obtainable. For the case of a lossless medium, since $\hat{S}(\omega)$ is unitary, we have

$$\hat{R}^A(\omega) = \hat{R}(-\omega) \quad \text{and} \quad R^A(t) = R(-t) \tag{6.14}$$

so that this additional information is redundant.

The layer-peeling method can now be used for the asymmetric two-component system by noting that, at point x, the anticausal waves $W_L^A(x, t)$ and $W_R^A(x, t)$ have the form

$$\begin{aligned} W_L^A(x,t) &= \delta(x+t) + w_L^A(x,t)u(-x-t), \\ W_R^A(x,t) &= w_R^A(x,t)u(-x-t) \end{aligned} \tag{6.15}$$

and that

$$k^A(x) = 2\omega_R(x, -(x+)). \tag{6.16}$$

Therefore, by using the system (6.9) with the relations (3.3) and (6.16) to propagate both the causal and anticausal pairs of waves $\{W_R(x, t), W_L(x, t)\}$ and

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

$\{W_L^A(x, t), W_R^A(x, t)\}$ simultaneously, we can recover both $k(\cdot)$ and $k^A(\cdot)$ in a sequential way. The Riccati equations for

$$\hat{R}(x, \omega) = \frac{W_L(x, \omega)}{W_R(x, \omega)} \quad \text{and} \quad \hat{R}^A(x, \omega) = \frac{W_R^A(x, \omega)}{W_L^A(x, \omega)} \tag{6.17}$$

are

$$\frac{d}{dt}\hat{R}(x, \omega) = 2j\omega\hat{R}(x, \omega) + k^A(x)\hat{R}(x, \omega)^2 - k(x), \tag{6.18}$$

$$\frac{d}{dt}\hat{R}^A(x, \omega) = -2j\omega\hat{R}^A(x, \omega) + k(x)\hat{R}^A(x, \omega)^2 - k^A(x). \tag{6.19}$$

These equations can be propagated recursively by using the relations

$$\begin{aligned} &\lim_{\omega\to\infty} 2j\omega\hat{R}(x, \omega) = k(x), \\ &\lim_{\omega\to\infty} -2j\omega\hat{R}^A(x, \omega) = k^A(x) \end{aligned} \tag{6.20}$$

which have the effect of coupling (6.18) and (6.19).

An integral equations based solution of the above problem can be found in Ablowitz and Segur [20].

7. Conclusions. In this paper we have presented differential inversion methods for identifying various parametrizations of lossless and nonlossless scattering media. These methods were also related to the classical approaches of Marchenko, Gelfand-Levitan and Krein which are based on solutions of Fredholm integral equations. Crucial in all the developments was the assumption that the given scattering data is noise-free; therefore the methods presented are *exact inversion algorithms.* An analysis of numerical conditioning, noise propagation and some ways to incorporate prior information on medium properties in the inversion algorithms, for the case when the data is corrupted by additive noise, has recently been completed [51].

The results described in this paper could be extended in several ways. One of these would be their use for the propagation of solutions of certain nonlinear differential equations by the inverse scattering transform [19], [20]. Also, our analysis has been restricted to physical processes described by second-order differential equations. It would be interesting to generalize the differential approaches discussed in this paper to study of inverse problems for more complex physical structures, described for example by general Hamiltonian systems.

REFERENCES

[1] L. D. FADDEEV, *The inverse problem of quantum theory of scattering,* J. Math. Phys., 4/1 (1963), pp. 72-104.

[2] K. CHADAN AND P. C. SABATIER, *Inverse Problems in Quantum Scattering Theory,* Springer-Verlag, New York, 1977.

[3] F. DYSON, *Old and new approaches to the inverse scattering problem,* in Studies in Mathematical Physics, Essays in Honor of V. Bargmann, Princeton Series in Physics, Princeton, NJ, 1976, pp. 151-167.

[4] J. A. WARE AND K. AKI, *Continuous and discrete inverse scattering problems in a stratified elastic medium* I: *plane waves at normal incidence,* J. Acoust. Soc. Amer., 45 (1969), pp. 911-921.

[5] L. WOHLERS, *Lumped and Distributed Passive Networks,* Academic Press, New York, 1969.

[6] B. GOPINATH AND M. M. SONDHI, *Inversion of the telegraph equation and the synthesis of nonuniform lines,* Proc. IEEE, 59 (1971), pp. 383-392.

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

[7] B. GOPINATH AND M. M. SONDHI, *Determination of the shape of the human vocal tract from acoustical measurements*, Bell Syst. Tech. Journal, 49 (1970), pp. 1195-1214.

[8] H. J. LANDAU, *The inverse problem for the vocal tract and the moment problem*, SIAM J. Math. Anal., 14 (1983), pp. 1019-1035.

[9] E. A. ROBINSON, *Dynamic predictive deconvolution*, Geophys. Prospecting, 23 (1975), pp. 779-797.

[10] ———, *Spectral approach to geophysical inversion by Lorentz, Fourier and Radon transforms*, Proc. IEEE, 70 (1982), pp. 1039-1054.

[11] V. BARDAN, *Comments on dynamic predictive deconvolution*, Geophys. Prospecting, 25 (1977), pp. 569-572.

[12] G. ARSENE, Z. CEAUŞESCU, F. POTRA AND D. TIMOTIN, *A new algorithm for detecting reflection coefficients in layered media*, Annal. Geophysicae, 1, 4-5 (1983), pp. 285-290.

[13] J. G. BERRYMAN AND R. R. GREENE, *Discrete inverse methods for elastic waves in layered media*, Geophysics, 45 (1980), pp. 213-233.

[14] I. M. GELFAND AND B. M. LEVITAN, *On the determination of a differential equation from its spectral function*, Amer. Math. Soc. Transl., series 2, 1 (1955), pp. 253-304.

[15] Z. S. AGRANOVICH AND V. A. MARCHENKO, *The Inverse Problem of Scattering Theory*, Gordon and Breach, New York, 1963.

[16] M. G. KREIN, *On a method for the effective solution of the inverse boundary value problem*, Dokl. Akad. Nauk SSSR, 94 (1954), pp. 987-990. (In Russian.)

[17] I. KAY AND M. E. MOSES, *Inverse Scattering Papers*: 1955-1963, Mathematical Sciences Press, Brookline, MA, 1982.

[18] L. D. FADDEEV, *Properties of the S-matrix of the one-dimensional Schrödinger equation*, Amer. Math. Soc. Transl., series 2, 65 (1967), pp. 139-166.

[19] V. E. ZAKHAROV AND P. B. SHABAT, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Sov. Phys. JETP, 34 (1972), pp. 62-69.

[20] M. J. ABLOWITZ AND H. SEGUR, *Solitons and the Inverse Scattering Transform*, SIAM Studies 4, Society for Industrial and Applied Mathematics, Philadelphia, 1981.

[21] M. S. HOWARD, *Inverse scattering for a layered acoustic medium using the first order equations of motion*, Geophysics, 48 (1983), pp. 163-170.

[22] P. DEIFT AND E. TRUBOWITZ, *Inverse scattering on the line*, Comm. Pure Appl. Math., 32 (1979), pp. 121-251.

[23] W. SYMES, *Stable solution of the inverse reflection problem for a smoothly stratified medium*, SIAM J. Math. Anal., 12 (1981), pp. 421-453.

[24] F. SANTOSA AND H. SCHWETLICK, *The inversion of acoustical impedance profile by methods of characteristics*, Wave Motion, 4 (1982), pp. 99-110.

[25] M. M. SONDHI AND J. R. RESNICK, *The inverse problem for the vocal tract: numerical methods, acoustical experiments and speech synthesis*, J. Acoust. Soc. Amer., 73 (1983), pp. 985-1002.

[26] K. P. BUBE AND R. BURRIDGE, *The one-dimensional inverse problem of reflection seismology*, SIAM Rev., 25 (1983), pp. 497-559.

[27] I. SCHUR, *Uber Potentzreihen, die im Innern des Einheitskreises Beschrankt Sind*, J. Reine Angew. Math., 147 (1917), pp. 205-232.

[28] N. I. AKHIEZER, *The Classical Moment Problem*, Hafner, New York, 1965.

[29] P. DEWILDE, A. C. VIEIRA AND T. KAILATH, *On a generalized Szego-Levinson realization algorithm for optimal linear predictors based on a network synthesis approach*, IEEE Trans. Circuits and Systems, CAS-25 (1978), pp. 663-675.

[30] P. DEWILDE, J. T. FOKKEMA AND I. WIDYA, *Inverse scattering and linear prediction, the time continuous case*, in Stochastic Systems: Mathematics of Filtering and Identification Applications, D. Reidel, Dordrecht, 1981.

[31] T. KAILATH, L. LJUNG AND M. MORF, *Generalized Krein-Levinson equations for efficient calculation of Fredholm resolvents of nondisplacement kernels*, in Topics in Functional Analysis, essays in honor of M. G. Krein, Advances in Mathematics Supplementary Studies, 3, Academic Press, New York, 1978, pp. 169-184.

[32] T. KAILATH, B. C. LEVY, L. LJUNG AND M. MORF, *The factorization and representation of operators in the algebra generated by Toeplitz operators*, SIAM J. Appl. Math., 37 (1979), pp. 467-484.

[33] B. D. O. ANDERSON AND T. KAILATH, *Fast algorithms for the integral equations of the inverse scattering problem*, Internat. J. Eqns. and Operator Thy., 1 (1978), pp. 132-136.

[34] I. GOHBERG AND I. KOLTRACHT, *Numerical solutions of integral equations, fast algorithms and the Krein-Sobolev equation*, The Weizmann Institute of Science, Rehovot, Israel, preprint, 1983.

[35] H. DYM AND H. P. MCKEAN, *Gaussian Processes, Function Theory and the Inverse Scattering Problem*, Academic Press, New York, 1976.

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms

[36] R. G. NEWTON, *Inversion of reflection data for layered media: a review of exact methods*, Geophys. J. Royal Astr. Soc., 65 (1981), pp. 191-215.

[37] J. D. MARKEL AND A. H. GRAY, *Linear Prediction of Speech*, Springer-Verlag, New York, 1978.

[38] R. CARROLL AND F. SANTOSA, *Scattering techniques for a one-dimensional inverse problem in geophysics*, Math. Meth. Appl. Sci., 3 (1981), pp. 145-171.

[39] E. A. ROBINSON AND S. TREITEL, *Geophysical Signal Processing*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[40] R. BURRIDGE, *The Gelfand-Levitan, the Marchenko and the Gopinath-Sondhi integral equations of inverse scattering theory regarded in the context of inverse impulse response problems*, Wave Motion, 2 (1980), pp. 305-323.

[41] R. REDHEFFER, *On the relation of transmission-line theory to scattering and transfer*, J. Math. Phys., 41 (1962), pp. 1-41.

[42] B. GJEVICK, A. NILSEN AND J. HOJEN, *An attempt at the inversion of reflection data*, Geophys. Prosp., 24 (1976), pp. 492-505.

[43] C. L. PEKERIS, *Direct method of interpretation in resistivity prospecting*, Geophysics, 5 (1940), pp. 31-42.

[44] O. KOEFOED, *The application of the kernel function in interpreting geoelectrical resistivity measurements*, Geoexploration Monographs, Series 1, No. 2, Gebruder Borntraeger, Berlin-Stuttgart, 1968.

[45] V. BELEVITCH, *Classical Network Theory*, Holden-Day, San Francisco, 1968.

[46] R. BELLMAN AND G. M. WING, *An Introduction to Invariant Imbedding*, John Wiley, New York, 1975.

[47] G. L. LAMB JR., *Elements of Soliton Theory*, John Wiley, New York, 1980.

[48] A. M. BRUCKSTEIN AND T. KAILATH, *Spatio-temporal scattering and inverse problems*, ISL Report, Stanford Univ., Stanford, CA, 1983.

[49] M. JAULENT, *The inverse scattering problem for LCRG transmission lines*, J. Math. Phys., 23 (1982), pp. 2286-2290.

[50] A. YAGLE AND B. C. LEVY, *The Schur algorithm and its applications*, LIDS Report, Massachusetts Institute of Technology, Cambridge, 1983.

[51] A. M. BRUCKSTEIN, I. KOLTRACHT AND T. KAILATH, *Inverse scattering with noisy data*, preprint, Stanford Univ., Stanford, CA, 1984.

This content downloaded from 132.68.46.14 on Fri, 15 Dec 2017 16:58:00 UTC
All use subject to http://about.jstor.org/terms